Self-organised critical control in human behaviour

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1. Introduction

As the top level of the central nervous system (CNS), the brain regulates virtually all human activity. Therefore, the brain may be considered a controller. This high-level view allows a comparison with artificial control systems which has revealed some similarities but also huge differences. These differences lead to different strengths and weaknesses for humans and engineering systems even at the level of very fundamental unconscious functions. For example, the control of movements may appear effortless, but trying to construct robots with the same skills reveals the high difficulty of this task. Furthermore, humans often adapt quickly to changing situations. Rapid adaptation has turned out to be difficult to achieve in robotics, too. Accordingly, examinations of the underlying neuronal structure have revealed a high degree of complexity. This has so far prevented researchers to fully understand the control mechanisms used by the brain and their biological implementations.

One of the difficulties in understanding how the brain works lies in the interaction between processes happening on many different levels. From single molecules to whole organisms, from fractions of a second to years – interacting structures on different scales distributed over many orders of magnitude in space and time are inherent in biology.

A striking property of complex systems in general are large-scale collective behaviours. Such large events are of high interest because they often represent crises like natural catastrophes, social upheaval, economic crashes or the failure of engineering systems. They also provide potential insight in the underlying structures that may be difficult to observe.

A well accessible field in brain research is the investigation of human behaviour. When humans perform closed loop control tasks like in upright standing, while balancing a stick or controlling a virtual target on a computer screen, their behavior exhibits large, non-Gaussian fluctuations similar to physical systems near a critical point \[2,3,7\]. The origin of these fluctuations is not known. In artificial control systems, engineers would most likely try to avoid critical fluctuations. Recently, the consequences of rapid adaptivity for the dynamics of a controller have been investigated. Optimal on-line adaptation of a controller using finite memory has been shown to generically lead to critical noise amplification. This process may be a source for the observed fluctuations \[5\].

This thesis focuses on the insights into human control strategies that can be gained through the investigation of these fluctuations. This is done in the framework of critical control. First, the reader is introduced to critical phenomena, human motor control and human balancing behaviour as far as necessary for the
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following considerations. Due to the size and complexity of these fields, providing all necessary basics and more than a brief overview lies beyond this work’s scope. Because this is a diploma thesis in theoretical physics, basic knowledge of statistical physics and probability theory are assumed. A comprehensive introduction to critical phenomena that also covers these topics has been written by Sornette [1]. In the next chapter, the minimal model for critical control as considered by Pawelzik et. al. [5,6] is derived. In the third chapter, existing data for a virtual balancing task obtained by Markus Riegel is put through a new, optimised and more comprehensive analysis and is compared to the basic model. The fourth chapter deals with extensions of the discrete model and their potential to generate properties found in the experimental data. Most of the results up to this point including parts of the new data analysis and modelling work performed for this thesis are published in [8]. Chapter five deals with the generalisation of the discrete-time model to a continous-time version. Statistical properties of the time series as well as qualitative features of the observed fluctuations on short time scales are compared with the experimental data. Also, the amount of random variability in the model is compared with the variability observed in the experimental time series. Possible further extensions needed for the application of critical control to systems with long-range correlations in time like financial markets are briefly discussed.
1.1. Critical phenomena

1.1.1. Critical points and phase transitions

A great diversity of length scales exist in the structure of the world. The success of almost all physical theories lies in isolating a limited range of length scales. For example, when throwing a ball its trajectory can be calculated with high precision without knowing the configuration of all atoms it consists of.

For certain phenomena, however, many scales of length contribute with equal importance. At the critical point, the largest fluctuations become infinite, but the smaller ones still make essential contributions to the systems behaviour. Consider, for example, a material that is ferromagnetic at low temperatures and paramagnetic at high temperatures. The materials magnetisation is the result of a competition between order and disorder. The alignment of magnetic dipoles in the material through interactions lowers the systems energy while thermal fluctuations destroy these correlations. At high temperatures, the system is globally disordered and only small clusters of aligned dipoles exist. At low temperatures, the system consists of ordered clusters, which are however not perfectly aligned relative to each other and may contain smaller clusters. The system can have a permanent macroscopic magnetisation. At the Curie temperature, the whole system forms one infinite cluster that contains smaller clusters which again contain smaller clusters and so on. This picture reminds of a Russian doll. The system is called scale-free. This fractal structure may exist between the atomic level and the size of the volume that is filled with the considered material. At the curie point, the magnetic susceptibility grows to infinity according to a power law. Fluctuations at the smallest clusters may only propagate to close neighbours, but because of the fractal clustering these fluctuations may grow up to the macroscopic level.

Critical phenomena include power law scaling relations that extend over several orders of magnitude and divergences of some quantities following power laws described by critical exponents. Usually, they originate in the divergence of a spatial or temporal correlation length which is the natural scale of the systems physical structure. In equilibrium systems, the critical point at phase transitions can only be reached by tuning a control parameter precisely. In the above example, this means adjusting the temperature to be exactly the Curie temperature.

1.1.2. Self-organised criticality

Following the above considerations, the widespread occurrence of persistent critical phenomena in nature still cannot be explained sufficiently. Many spatially extended objects like coastlines, clouds or vessels approximate fractal structures. Also, fluctuations with power law scaling in their power spectra — called $1/f$-noise — are found in many systems like the flow of rivers or the luminosity of stars. Many complex systems also exhibit fluctuations which themselves obey scaling relations. Such non-Gaussian fluctuations of state variables $x$ with distributions $P(x)$ are well described by $P(x) \propto |x|^{-\delta} \ (\delta > 1)$ at least for large
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Magnitudes of \( x \). Examples include sizes of avalanches of granular matter, the distribution of earthquake magnitudes (Gutenberg-Richter law) and stock-market log-return fluctuations \([1]\). Recently, theoretically predicted power law distributions \([38]\) were experimentally found \([19]\) also in the firing behavior of neural populations in cortical tissue.

The first explanation for the robust emergence of critical phenomena without fine tuning of parameters was given by self-organised criticality (SOC). In 1987, Bak et. al. \([11]\) demonstrated, that certain classes of dynamical systems have a critical point as an attractor. SOC is typically observed in slowly driven non-equilibrium systems with extended degrees of freedom and non-linear interactions.

An introductory example is the formation of a sandpile. The system is externally driven by adding sand grains one after another. If the slope of the surface is too large, the pile will collapse. When the average slope reaches a critical value, the system is barely stable with respect to small perturbations. In this state, avalanches of many different sizes are possible. The \( 1/f \)-noise is the dynamical response of the sandpile to small random perturbations.

For many SOC models, the distribution of fluctuations can be mapped to the first return time of a random walk \([1]\). These distributions are characterised by a scaling exponent of value \( 3/2 \). This is already a hint on simpler mechanisms that can generate critical behaviour.

1.1.3. Phase-space analysis

The transition of a system between two states is in mathematics called a bifurcation. In dynamical systems, this means that a small change to certain parameter values cause sudden qualitative change in the systems dynamical behaviour. Near an equilibrium (or fixed) point for a dynamical variable (the so-called order parameter), many complex systems can be simplified in such a way, that its behaviour is characterised by a single bifurcation parameter. At a critical value of this parameter, the stability of the fixed point changes. A critical phase transition is called a “supercritical” bifurcation. Fig. 1.1 shows a bifurcation of the order parameter from a reference value zero to a regime with two possible values represented by two branches. As the system approaches the critical point from the stable regime, fluctuations become critically amplified. At the critical point, the average standard deviation of the fluctuations of the order parameter diverges. This situation corresponds to the divergence of the susceptibility as discussed above. Very close to the critical point the largest possible fluctuations may be limited by nonlinear saturation \([1]\).

Systems that switch between qualitatively different kinds of oscillations in an apparently random way are called intermittent. Such systems can be constructed around (quasi-) invariant objects in a systems phase space near to which the system will tend to spend long times. One example is an invariant unstable object combined with a reinjection mechanism driving the system back to the
instability when it is far away from it [12]. The mechanism is depicted in Fig. 1.2. Some multiplicative noise processes can produce a special kind of intermittency with power law distributed fluctuations. These processes can be completely linear and produce critical behaviour for relatively mild conditions for the driving noise. Especially, they do not require the exact tuning of a bifurcation parameter to a critical value. Such a process is discussed below.

1.1.4. The Kesten process

Consider the linear first order recurrence relation

\[ y_{t+1} = \alpha_t y_t \]  \hspace{1cm} (1.1)

where \( \alpha_t \) is a stochastic variable with probability distribution function (pdf) \( P(\alpha) \). The logarithm of \( y_t \) is the sum over \( t \) independent identically-distributed (iid) variables. As long as the central limit theorem is valid for \( P(\alpha) \), this sum converges to a gaussian distribution. This means, that \( y \) is distributed according to the log-normal distribution. To turn \( P(y) \) into a power law we need to make sure, that (1.1) contracts on average and is repelled from the origin. While keeping the random map linear, this is fulfilled for a negative logarithmic growth rate \( \langle \ln |\alpha| \rangle < 0 \) by introducing an additive noise term \( \beta_t \). The resulting process

\[ y_{t+1} = \alpha_t y_t + \beta_t \]  \hspace{1cm} (1.2)

is called the Kesten process. The essential results can be summarised as:

- If \( \alpha_t \) and \( \beta_t \) are i.i.d. real-valued random variables and if \( \langle \ln |\alpha| \rangle < 0 \), then \( y_t \) converges in distribution and has a unique limiting distribution \( P(y) \).
- If, additionally, \( \beta_t/(1-\alpha_t) \) is nondegenerate and if there exists a \( \mu > 0 \) with
  1. \( 0 < \langle |\beta|^\mu \rangle < +\infty \),
  2. \( \langle |\alpha|^\mu \rangle = 1 \) and
  3. \( \langle |\alpha|^\mu \ln^+ |\alpha| \rangle < +\infty \)

then the limiting distribution for \( y_t \) is for large \( y_t \) asymptotic to \( P(y) \propto y^{-\delta} \) as \( y \to \infty \) with \( \delta = \mu + 1 \) (11), with corrections from [9]).

While the multiplicative term causes the process to contract on average, fluctuations across the stability boundary \( \ln |\alpha| = 0 \) cause intermittent amplification. This process is similar to another form of intermittency, called on-off intermittency. For these systems, a time-dependent bifurcation parameter is varied repeatedly across its critical value. This causes intermittent fluctuations in a second dynamical variable. An example for on-off intermittency is a random or chaotically driven logistic map. The main difference between the intermittent behaviour of the Kesten process and on-off intermittency as reported in [12, 10].
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is, that these models show nonlinear saturisation. This prevents the appearance of large fluctuations and therefore of power law tails in the distributions \[13\]. Saturation is necessary because the intermittent regime in on-off intermittency is associated with the systems fixed points unstable regime. Fig. 1.3 shows the time series and distribution functions\(^1\) for different processes \(y\): A: gaussian noise, B: a logistic map with parametric noise exhibiting on-off intermittent bursting as considered in \[10\] and C: the kesten process. While the latter doesn’t require a perfect tuning onto the stability boundary to yield observable power laws in the distribution of \(y\), the logarithmic growth rate still has to be adjusted to be negative and close to 0. Reducing it from 0 to \(-0.1\) causes the exponent \(\delta\) to change from 2 to 4 as shown in Fig. 1.3 D. For more details on the applied methods, see Sec. 3.2.

The Kesten process serves as a starting point for a model with power law distributed fluctuations. The next section gives an overview over basic principles in motor control to provide the fundamentals for the constructions of such a model in the context of balancing behavior.

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\(^1\) The complementary cumulative distribution function \(F_c(Y) = P(Y > Y')\) corresponds to the probability that a process \(Y\) takes a value larger than a real number \(Y'\). Here, \(Y = |y|\).
1.2. Motor control

Influencing the behaviour of dynamical systems to achieve a desired goal is dealt with by control theory \cite{14}. The desired state or output of a system is called the reference. The controller manipulates one or more variables of a system to make the output follow the reference. In the following, we mainly investigate how humans perform balancing-like tasks. Therefore, it seems appropriate to focus on control theory applied to human behaviour, especially to arm movements.

Human motor skills are skills that require utilising the skeletal muscles. In this context, motor means muscles that create movement by applying forces to bones and joints. A goal may be for example pointing to a target. The subjects CNS acts as a controller by influencing the contraction of said muscles as to change the position of the arm to match the reference position. That is, a position in which the subjects arm points to the target.

Many aspects of human control behaviour are still unknown. However, unifying principles in computational motor control have emerged that provide a theoretical framework for movement neuroscience. This introduction will give a brief summary of the current opinion according to \cite{16,15,17} as far as the proposed mechanisms are of interest for this thesis.

Fig. 1.4 shows the classical closed-loop controller. Such a controller uses feedback to control the output state. This creates a control loop: The input influences the output which is measured using sensors, fed back and used as an input, thereby closing the loop. Control without feedback is called open-loop.

For human control behaviour, the situation becomes a lot more complicated. The output state may for example be the set of activations of muscles or the position and velocity of the hand. This state changes continuously during movement. Other parameters like physical properties of the body or the identity of a manipulated object may change on other timescales or discretely. Furthermore, sensory feedback and the planning and execution of motor commands involve delays that have to be incorporated into the control strategy. To do this, the CNS combines the actual sensory information about the world with an internal representation of the causal relationship between actions and their consequences. Such a representation is called a forward model.

Successful prediction improves control by moving delays out of the feedback loop. For example, when tracking a target on a touch screen with a finger, hu-
mans predict the movement of the target [13]. For unpredictable changes in the
movement of the target, it takes around 250 ms to the onset of corrective finger
movements and several hundred milliseconds more to reacquire the target.

In other experiments, tracking non-moving targets when they appear by eye,
head and hand has been investigated. Eye movement and electromyographic
(EMG) activity in the biceps and neck muscles are initiated with a delay of 200-
220 ms while head and arm movements start 300-350 ms after the target appears
[24]. In numerous studies, simple reaction times for detecting just the presence
of an expected visual stimulus of 180-200 ms on average have been reported [27].

The human sensorimotor loop can be divided
into three stages shown in Fig. 1.5. A motor com-
mand is generated by the CNS based on its informa-
tion about the task, context and output state. To do
this, an inverse internal model maps the desired con-
sequences of a control action to an appropriate com-
mand. Execution of the motor command changes
the output state. These consequences are predicted
by a forward dynamic model. The loop gets closed
by the sensory feedback caused by the changed out-
put state. Internally, a forward sensory model pre-
dicts this new feedback based upon the output of the
forward dynamic model.

The CNS uses different coordinate systems for
different computational tasks. One reason why this
is necessary is the following problem: Consider the
approximately 600 muscles in the human body as
being either contracted or relaxed. Even in this sim-
plified situation, there are $2^{600}$ possible motor acti-
vations. A look-up-table from motor activations to
sensory feedback alone would need more entries than a list of all atoms in the
universe. To make control a feasible problem, the information about the state of
the system necessary for a given task needs to be represented in a far lower dimen-
sional space. For example, eye-centred spacial coordinates have been proposed to
be used for working memory of eye movements and for ongoing or intended
arm movements. Transformation to motor coordinates may then be performed
after response selection. Fig. 1.6 shows the inverse model for multi-joint arm
movements in more detail. First, a trajectory is planned in spacial hand coor-
dinates. Second, the joint coordinates of the desired position of the whole arm
are found using inverse kinematics. Third, the transformation into motor com-
mands requires the calculation of joint torques. This is done using inverse dynamics
or possibly using equilibrium point models which rely on spring-like proper-
ties of muscles and reflex loops. These loops between muscles and neurons in
the spinal cord adjust the contraction of muscles much faster than signals can be
1.3. Balancing

1.3.1. Postural control

Quiet standing may appear to be static, but it is a complex dynamical control task. The human upright posture is unstable and requires an ongoing active balancing which leads to body sway. Numerous studies have investigated the movement of the center of pressure under the feet of humans standing on force platforms. Body movements have also been measured using ultrasonic or infrared reflectors. [25]

The fluctuations found in human postural sway have been found to show statistical features of a correlated random walk for short timescales. At a correlation time of approximately one second, a crossover to an anticorrelated random walk occurs [19]. For very long timescales, the timeseries become uncorrelated [26].

Models of postural sway as an overdamped inverse pendulum have been shown to reproduce these experimental features. Initially, the different scaling regimes have been attributed to open- and closed-loop control. A possible model includes a controller that alternates between two springs based on a threshold [23]. Later, it has been shown, that different scaling regimes can arise even in linear control without a delay due to dynamic properties of the body and the controller [21]. Strong noise can of course destroy correlations. For example, noise-induced transitions can destroy otherwise stable limit cycles [26].
1.3.2. Stick balancing

Another approach to investigate the noisy fluctuations found in human control behaviour has been adopted by Cabrera and Milton [2][3][4]. They investigated humans balancing sticks on their fingertips while sitting on a chair. The movements of both endpoints of the sticks in three dimensions were captured by motion capturing cameras. The analysis has been focused on the cosine of the vertical displacement angle

$$\cos(\theta) = 1 - \frac{Y}{L}.$$ 

This equals one minus the difference $Y$ between the projection of the endpoints in the horizontal plane normalised by the stick length $L$. Fig. 1.7 shows a sketch of the experimental set-up. The published plot of the time-series in $\cos(\theta)$ shows non-Gaussian fluctuations. The following results for these fluctuations have been reported:

- periods of small fluctuations alternate with shorter periods containing larger fluctuations;

- the baseline for the fluctuations corresponds to a slight deviation from the upright position of the stick;

- the power spectrum for the fluctuations contains two scaling regimes. The first one exhibits an exponent close to $1/2$. The second regime starts above one Herz. It has a scaling exponent close to 2.5;

Bursting behaviour can occur in systems with on-off intermittent behaviour as discussed in Sec. 1.1.4. Because the observed fluctuations also show other features that are observed in such systems, the authors suggest that human balancing behaviour shows on-off intermittency. For further statements regarding the intermittency type and laminar phases, see the remark on the next page.

The most important conclusion that is reported is, that the system is tuned very close to a stability boundary with parametric noise causing fluctuations across the boundary. To model this behaviour, the authors choose an overdamped inverse pendulum. The controller applies a force proportional the value of the angle $\theta(t - \tau)$ delayed by a reaction time $\tau$. The proportionality parameter is then perturbed by multiplicative noise. The overdamped inverse pendulum model shows scaling behaviour in the power spectrum and in the laminar phases and bursting fluctuations for carefully adjusted parameters. As stated above, this can be achieved with a variety of models and this particular model has some drawbacks that will be discussed at the end of this section.

No mechanism has been proposed for the self-tuning of the system to the critical point. Such a mechanism will be of central concern in the models investigated in the following chapters. No refinement of the inverse pendulum model will be made, since the experimental data used for comparison has been obtained in a virtual balancing task. In this task, the dynamics displayed on the screen have
not been based on a mechanical system. Therefore, control strategies employed by the brain in these two different experimental paradigms may be different. However, the observed fluctuations show very similar statistical properties. Both problems may hence be solved by the brain using similar mechanisms.

Remark on intermittency types

Carbera and Milton report a scaling of the laminar phases with an exponent of $3/2$. Together with a scaling behaviour in the power spectrum with exponent $1/2$, this is characteristic for on-off intermittency. However, it is necessary to investigate not only the measured properties, but also the detailed mechanism for the intermittent behaviour to determine its type \[10\]. Therefore, a classification of intermittency types lies beyond the scope of this work.

Note, however, that the power spectra for balancing tasks show not one, but two scaling regimes. Both exponents are not universal, but parameter dependent in the mechanical delayed feedback model as well as in the models considered in this thesis. Also, the exponents in the virtual balancing time-series are not universal. For the experimental and simulated time-series in this thesis, the laminar phases seem to obey a scaling behaviour as well as an exponential cutoff. As already mentioned in \[1.1.4\], power law distributed fluctuations of a dynamical variable hint on intermittency caused by a Kesten-like process which is very similar, but not identical to on-off intermittency.

As a last remark note that Carbera and Milton consider the times between threshold crossings in the direction of the upward position of the stick to be the laminar phases. Usually, the lengths of the phases below a threshold are called the laminar phases. However, own very brief numerical investigations didn’t reveal big differences between these two definitions (not shown).

Discussion of the overdamped inverse pendulum model

Time-delayed feedback control without strong damping cannot be stable when only proportional feedback is used \[22\]. For awake humans, muscles and local reflex loops overdamp joint movements. Therefore, overdamping makes sense in postural control models. When balancing a stick, only one end of the stick is controlled so it is not clear why the uncontrolled stick should behave like an overdamped pendulum. Without overdamping, at least the derivative of the controlled variable has to be used additionally by the controller. It has been supposed, that humans also use the integral over past values \[21\]. Also, there seems to be a consensus in movement neuroscience, that humans use predictive control (see section \[1.2\]).

Another problem is, that a classical controller is either stable or unstable. This means, that the controlled variable either converges to zero or diverges. If the system is not completely overdamped, decaying or growing oscillations or an unstable limit cycle for oscillations exactly on the boundary are possible, too. For very slow convergence due to strong damping and high multiplicative noise
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levels it is possible, that the system shows irregular bursting over a long period of time. This leads Cabrera and Milton to assume to have demonstrated, that the system shows on-off intermittency although it never combines instability at the origin and stability at large distances.

Reported reaction times as low as 70ms measured by comparing the correlation between both ends of the stick also seem extremely low compared to the literature [12]. These short delays have been used in the model. If these correlations correspond to real corrective movements and are not artefacts, then they are most probably based on predictions and not directly on delayed feedback.

The authors argue, that the main question is why the stick falls down at all, given that a classical controller can stabilise an inverse pendulum. They suggest, that the system is tuned to a stability boundary because a random walk around the origin corresponds to the stick being stabilised on average. The self-similar nature of the random walk also causes threshold crossings from larger distances to smaller ones on all time scales which the authors interpret as corrective movements faster than the reaction time.

Noise can stabilise metastable or unstable systems and therefore, it is suggested, that this is one of the reasons for high noise levels in human control behaviour. However, stabilising by noise is usually achieved through small amounts of additive noise (see, for example, [23]). The authors don’t give an example for stabilisation by multiplicative noise, and they don’t show, that their model is somehow stabilised by noise.

The distribution of fluctuations for stick balancing have only been considered for velocity increments. This has been done by using a theorem on return probabilities to the origin valid for special processes called Levy flights. Since only the center of the distribution is used, conclusions can be drawn only given that the high fluctuation regime is dominated by the same mechanism as the one for small fluctuations and that the considered process is a Levy flight. For stock market fluctuations, this method is known to yield different scaling exponents than optimal estimators for the tails of the distribution [31]. This has been confirmed for the measured and simulated time-series used throughout this thesis and therefore, the probability of return to the origin will not be considered any further. Since large fluctuations are of most concern when they appear in the controlled coordinate itself, velocity changes are not considered either. However, investigations using this method yield results for the experimental and simulated time-series considered in this thesis that are consistent with those reported in [3].

\[ \text{In [2], [18] is cited where much higher reaction times have been measured. In this work three different models for tracking have been proposed. One of the models contains three different delays for reaction on different changes in the movement of a target and the best overall fit to measured data is obtained when one of these delays is as low as 115ms. This doesn’t fit the onset of corrective movements very well and the authors of the paper suggest, that this model is not as realistic as another one they consider and that delay times are in fact higher.} \]
2. A minimal model for critical control

As reported in the previous chapter, human balancing behaviour shows fluctuations usually observed in systems at critical points. In the following, we will derive a minimal model for a control system that has a critical point as an attractor.

A time-discrete random map

\[ y_{t+1} = \alpha y_t + \beta_t, \]  
(2.1)

defines a control problem where the dynamical variable \( y_t \) denotes the deviation of a system from the reference value at time \( t \) (\( t = 0, 1, 2, \ldots \)). \( \alpha \) is a system parameter unknown to the controller and assumed to be constant for at least some period of time. For \( \alpha > 1 \), the fixed point at the origin is linearly unstable. \( \beta_t \sim \mathcal{N}(0, \sigma^2) \) is a Gaussian random variable describing non-predictable fluctuations. Its variance \( \sigma^2 \equiv \text{const.} \) is a second hidden system parameter.

The controller is assumed to know the form of the dynamical equation (2.1). The control strategy involves computing an estimate \( \alpha_{t+1} \) of the unknown parameter \( \alpha \) from previous observations \( y_t, y_{t-1}, \ldots \) of the system. The deviation \( y_{t+1} \) from the reference at time \( t + 1 \) is predicted with a forward model of the simple form

\[ \hat{y}_{t+1} = \alpha_{t+1} y_t \]  
(2.2)

using the estimated parameter and the observed previous deviation \( y_t \). The controller tries to bring \( y_{t+1} \) to zero by simply removing the predictor \( \hat{y}_{t+1} \). When control is switched on, Eq. (2.1) is replaced by

\[ y_{t+1} = (\alpha - \alpha_{t+1}) y_t + \beta_t. \]  
(2.3)

Note that we inserted the explicit form of the predictor. This is convenient for the minimal model, but we will return to specifying the predictor in a separate equation when we extend the model later.

If the estimator \( \alpha_{t+1} \) fluctuates around the real parameter \( \alpha \), then in every timestep \( y \) is multiplied by an effective random variable \( (\alpha - \alpha_{t+1}) \) with some distribution \( P(\alpha') \). If the \( \alpha' \) are i.i.d. random variables, we recover the...
Kesten process [1,2]. The multiplicative noise term represents the fluctuations that enter the system through noise in the estimated parameter used by the controller.

To obtain critical fluctuations, \( \langle \ln |\alpha'| \rangle \) has to be slightly negative or exactly zero. This can be achieved by adding a bias to the estimation or by changing its variance. In either case, we need a mechanism for tuning the system to the critical point. This may also be seen as the minimal version of the unsolved tuning problem in the stick balancing model described in 1.3.2.

In the following, we employ a minimum mean squared error approach for the controller’s estimation of the control parameter \( \alpha \) i.e., \( \langle y^2 \rangle = \min \) where the brackets denote the expectation with respect to the additive noise. The expectation value of the squared deviation from the target is

\[
\langle y_{t+1}^2 \rangle = \langle (\alpha - \alpha_{t+1})y_t + \beta_t \rangle^2 \tag{2.4}
\]

For simplicity we here assume that only the observable values of \( y_t \) and \( y_{t-1} \) and the previous control actions are taken into account by the controller and we use

\[
\alpha = \frac{y_t - \beta_{t-1}}{y_{t-1}} + \alpha_{t-2} \tag{2.5}
\]

( obtained from rearranging (2.3) where we replaced \( t \) by \( t - 1 \) to eliminate \( \alpha \) in 2.4. Minimising Eq. 2.4 with respect to \( \alpha_{t+1} \) yields the optimal estimator \( \alpha_{t+1} \) from the two very recent observations:

\[
\alpha_{t+1} = \frac{y_t}{y_{t-1}} + \alpha_t. \tag{2.6}
\]

Eqns. (2.3) and (2.6) represent the dynamics of the basic adaptive control system as a 2-dimensional map. Using (2.3) to replace \( y_t \) in (2.6), the estimator can be written in the form

\[
\alpha_{t+1} = \frac{\beta_{t-1}}{y_{t-1}} + \alpha. \tag{2.7}
\]

This form of the equation shows that the estimated value of the system’s parameter becomes dominated by the noise \( \beta_t \) when \( y_t \) has small values. Here, the multiplicative noise term is the consequence of optimal short time parameter estimation.

The simple map (Eqns. (2.3) and (2.6)) generates time series with large fluctuations. Fig. 2.2A shows the time-series for 10^7 iterations of the system.

While this controller is optimal in the sense that it uses the minimum mean squared error estimate of the system’s parameter \( \alpha \) from two past observations, the large fluctuations indicate that the system is sub-optimal from a global perspective when the control is rather successful, i.e. when the magnitudes of \( y_t \) become small. Intuitively, for small amplitudes estimating the parameters doesn’t make sense because the dynamics is then dominated by the noise. In other words,
2 Minimal model

Figure 2.2: A: time-series of the minimal model ($\alpha = 2$, $\sigma = 2$).

B: thick line: complementary cumulative distribution function $F_c$ of the magnitudes $Y = |y|$ for the same model. Thin line: $F_c(Y_t / Y_{t-1})$ overlaps with $F_c$ for Gaussian noise ($\cdots$).

A controller with unlimited sensitivity will always run into the point where its estimate fits only the noise. In this sense the simple model system exhibits self-organized criticality.

Inserting (2.7) into (2.3) yields

$$y_{t+1} = -\frac{\beta_{t-1} y_t}{y_{t-1}} + \beta_t$$  \hspace{1cm} (2.8)

which represents the effective behaviour of the control system. This form shows the difference to the Kesten process: The multiplicative noise is in every timestep scaled with the reciprocal of the preceding two values of the process. Therefore, three subsequent values of $y$ are correlated. Also, there is only one process consisting of independent normal distributed random variables driving the system. The multiplicative variable consists of the previous value of the additive noise variable. Unfortunately, the analytical results for the Kesten process hold true only for independent random variables.

Nevertheless, a power law probability density with an exponent of $\delta = 2$ for the basic system for large fluctuations can be determined from an analytical treatment \[8\]. Fig. 2.2 B shows the complementary cumulative distribution function for the magnitudes $Y$ of $y$ (thick line) in double logarithmic coordinates. Above values of approx. $Y = 10$, $F_c(Y)$ is well described by the power law (short line). Analytically and numerically, $\delta$ is independent of $\alpha$ and $\sigma$. (2.8) shows, that the only characteristic scale in the system is the amplitude $\sigma$ of the gaussian driving noise.

Another interpretation of (2.8) is, that $y$ is the sum of two random variables, one of which is scaled by the ratio of the two preceding values $y$. Multiplying $Y$ with this ratio reveals a distribution (thin line in Fig. 2.2 B) that can be fit well by a gaussian with the same variance (dotted line). Further numerical analysis of the model can be found in the next chapter.
3. Data analysis and comparison

In this chapter, data sets from virtual balancing experiments provided by Markus Riegel are analysed. Several methods are introduced that allow a detailed comparison with simulated time-series.

3.1. A simple control experiment

Human control behaviour has been investigated in a simple balancing task [8]. A target and a cursor were presented on a computer screen. The x- and y-components of the target position $T$ moved according to

$$T_{i+1} = T_i + \alpha (T_i - M_i) + \beta_i, \quad i \in \{x, y\} \tag{3.1}$$

and the cursor position $M$ was controlled by the subjects using a computer mouse. On the left-hand side in Fig. 3.1, the experimental setup is depicted. The rectangle on the righ-hand side represents the scene displayed on the computer screen.

The task of the subjects was to stabilise the target by moving $M$ as close to $T$ as possible. This situation is for each component equivalent to the minimal control problem described above if the controller would move the cursor to the target prediction $M_{i+1} = T_i + \alpha_i (T_i - M_i)$ which defines his estimation of the control parameter $\alpha_i$. Subtracting $M_{i+1}$ on both sides in Eq. (3.1) then yields with $y_i = T_i - M_i$. Furthermore, it is assumed that the noise $\beta_i$ in the experiment is realised by noise inside the brain and the motor system controlling the hand guiding the mouse.

Seven subjects participated in the experiments, filled a questionnaire according to the ethical requirements of human experimentation, and declared their informed consent. The subjects were of ages 21 to 59, right-handed and one subject was female. During experiments, each subject was positioned 60 cm in front of the screen in a closed room without window, which was only weakly lighted.
from the back. During a trial, two squares were presented on the screen: a green one (M) for the mouse and a one red (T) for the target. Both squares were of 5 pixels side length. M was moved linearly proportional to the position of the subject’s hand using the mouse, limited by the screen border. The movement of the mouse was constrained to a low friction glass mouse pad of 25 cm $\times$ 30 cm size. 10 mm mouse movement corresponded to 445 pixels. Temporal resolution of the mouse was 2 ms and its spatial resolution corresponded to 1.41 pixels. Task was to keep M and T together as close as possible without one of them running out of the screen, a situation similar to balancing a stick on the tip of a finger. A trial was started by pressing a button on the keyboard and began with M and T being placed in the middle of the screen.

The parameter $\alpha$ was chosen larger than one. Thereby M represents an unstable fixed point of the dynamics of T. The larger $\alpha$, the more difficult was the task. Subjects 1, 2, and 3 performed the tasks with $\alpha = \text{const.}$, for the others $\alpha$ was randomly switched every second to a value in $\{3, 4, 5, 6\}$.

During a trial, the x- and y-positions of mouse and target were recorded as events with a frequency that was coupled to the refresh rate of 85Hz of the screen. A trial ended if either one of the points left the screen or the maximal recording time of 3 minutes had expired. Before the subsequent trial a break of 5 minutes allowed for relaxation. At least 10 trials per day were recorded. Experiments were authorized by the ethics committee of the University Bremen.

The experimental setup was prepared by Dr. Udo Ernst. The computer was a PC with 1800 Mhz and 512 MB working memory, a graphics card with a GeForce FX chip set and 64 MB memory, a USB computer mouse with 800 dpi resolution and a sampling rate increased to its maximum of 500 Hz, and a 19 inch color monitor with 85 Hz and resolution of 1280 $\times$ 1024 pixels. The program controlling the experiments was designed to nearly achieve real time processing. Operating system was Linux Fedora Core 3 (Version 11/2004) with real time capability provided by the RTAI 3.1 package [41]. Movement of target and cursor was controlled by a program written in C using OpenGL as a graphical frontend. Real time operation of this system was verified, occasional time errors were of the order of 2 ms.

3.2. Data analysis

3.2.1. Relating experimental to simulated time-series

Both, experimental and simulated data were analysed with several statistical methods. Because the experimental task was to align mouse and target on a two dimensional computer screen, the resulting data has two components for both, the positions of mouse and target. The differences

$$y^i_t = T^i_t - M^i_t, \quad i \in \{x, y\}$$

(3.2)
are the distances between mouse and target in the x- or y-direction on the screen in pixel units at time t. Since the axes of the screen represent an artificial coordinate system, it seems unlikely that the separate representation in x- and y position is of any meaning to the brain. For that reason, we focused on the Euclidean radial distance between mouse and target

\[ Y_t = \sqrt{(x_t - x)^2 + (y_t - y)^2}. \] (3.3)

The main difference between the distance and the differences in the components is, that \( Y \) is always positive. Otherwise, the radial distance and the absolute values of the differences in the components have almost identical statistical properties.\[28\]. Since the probability distributions of the components of the differences are symmetric, the positive part and the mirrored negative part of the difference distributions and the distributions of the absolute values of the distances overlap perfectly.

To compare the simulated time series with the experimental ones, the absolute values of \( y \) have been analysed analogously to \( Y \). In the following, we will refer only to the distances \( Y \) where for the simulated data \( |y| \) is implied.

### 3.2.2. Estimating scaling exponents

Estimating parameters of distributions of rare, extreme events in general is quite difficult and therefore has to be done carefully. The tail exponents were obtained for the probability distributions by using the maximum likelihood estimator for rank-ordered distances \( Y \) known as ‘Hill estimator’ \[1\]. This means, that the values of \( Y \) are sorted in descending order. The first rank is then the largest value of \( Y \) in the time series, the second rank the second largest value and so on. The self-similarity of fluctuations with a power law pdf with exponent \( \delta = \mu + 1 \) is expressed in the ratio of the probabilities for observing two values of \( Y \)

\[ \frac{p(Y_i)}{p(Y_j)} = \left( \frac{Y_i}{Y_j} \right)^{-\mu} \] (3.4)

which is also a power law. The estimator uses the first \( r \) ranks, which correspond to the largest events up to a cutoff rank \( r \). Smaller events which may belong to a different regime are discarded. The maximum likelihood estimator for \( \mu \) based on the ratio of the values \( Y_i \) at the considered ranks to the value \( Y_r \) at the cutoff rank is given by

\[ \mu = \left( \frac{1}{r} \sum_{i=1}^{r} \ln \left( \frac{Y_i}{Y_r} \right) \right)^{-1}. \] (3.5)

\[1\]This is consistent with simulations of the model in two independent dimensions (not shown). However, the magnitudes of the x- and y-components of the data sets are correlated for short times (<1s). Hence, comparing the radial distance with the one-dimensional model seems appropriate. In other studies, correlations between x- and y-components have also been found in two-dimensional tracking tasks even when the displayed dynamics changed only in one of the components \[18\].
This method takes into account the strong statistical fluctuations in the largest events and has therefore the advantage of being mathematically well founded, requiring only one cutoff and no binning.

The Hill estimator is sensitive to the cutoff because the beginning of the power law regime varies between trials and simulations with different exponents. Unfortunately, the onset of the scaling regime varies between different data sets as well as between different models and parameter sets. To improve robustness while minimising manual interaction, the following procedure was developed. The exponents $\mu_r$ for 100 different logarithmically spaced cutoff ranks are calculated. For time series of the order of $10^5$ to $10^6$ steps, like the experimental data sets and the simulations analysed for comparison, it turned out that the first 45 and the last 30 percent of the log of the range of the ranks can be regarded as not being well suited as a cutoff rank, which therefore has to lie within this range.

For the examination of the parameter dependence of the exponent in Chap. 4 and 5, the size of the time series has been increased as much as possible given the available computational resources. Here, the restrictions on the cutoff could be relaxed, since the criterion used to determine the cutoff was less probable to be fooled by random bumps or kinks.

The exponent $\mu = \mu_r$ of the fit with a cutoff rank

$$r_c = \text{argmin}_{r} \frac{1}{r} \sum_{i=1}^{r} Y_i^2 \left( \left( \frac{i}{r} \right)^{-1/\mu_r} - 1 \right)^2$$

in the relevant range and the least square error per rank is regarded as being optimal. This method has been tested on all data sets from whole days as well as from all single trials and from the simulated data. It turned out to provide reasonable results for all but few very crooked distributions. Another method tested was to search for the minimum of the slope of the estimated exponent when varying the cutoff. This method on average delivers results consistent with the first method, but since it acts only local it is less robust. Therefore, this method was only used for consistency checks and not for any of the results shown.

The rank ordered events can easily be plotted in the more familiar form of the complementary cumulative distribution function $F_c(Y)$. The value of this function is the probability to observe an event that is larger than the one specified in the argument. Therefore, $F_c$ can be obtained simply by dividing the rank of $Y$ by the total number of events.

3.2.3. Summary of the employed methods

To obtain better statistics, the repeated trials of one of the participants were combined into one time series with a length of some $10^5$ events. Results for this experimental time series and for the minimal model are presented in Fig. 3.2 and Fig. 3.3. The analysis contains four subplots as explained below. Fitted parts are drawn with thicker lines than extrapolations. Gaussians are dotted. Fits in sub-plots C and D have been obtained using the method of least square errors.
A. **The time series** $Y$ plotted over the time measured in seconds or steps.

B. **The complementary cumulative distribution function** $F_c(Y_t) = P(Y > Y_t)$ plotted in double logarithmic axes. The exponents $\delta$ of the tails of the probability densities of $Y$ have been calculated using the Hill estimator as described above. Note, that the exponent $\mu$ from the complementary cumulative distribution functions is related to the exponent $\delta$ in the pdf as $\mu = \delta - 1$.

C. **Power spectrum** of $Y$. To reduce noise, for the combined time series from one day, the time series is divided into 50 parts for each of which the power spectrum is computed individually. These 50 spectra are then averaged. The resulting spectrum has a lower noise level at the cost of raising the lowest frequency representable. Since the power spectrum is constant for low frequencies, the number of parts has been selected to obtain more information on the decay at high frequencies at the cost of the low frequencies. For the simulated time series of the discrete models, 200 of the power spectra parts are averaged. To reveal the scaling regimes, the axes are double logarithmic. Because the spectra are linearly spaced in the frequency domain, the averaged spectra have been logarithmically binned into 100 bins.

D. **Scaling of the variance** of the sums of subsequent values of $Y$. For i.i.d. random variables with finite variance, it scales with the number $n$ of summands as $n^{2H} = n^1$ (random walk), where $H$ is the Hurst exponent.

    Note, that the variance is only defined for processes $y$ with a pdf that decays faster than $|y|^{-3}$. This is trivially fulfilled if the pdf vanishes outside a finite region. Since the experimental dynamics were restricted to the screen, this is fulfilled for all experimental time-series. However, the minimal model described in the last chapter decays slower than necessary and is not restricted in the possible magnitudes of $Y$. Therefore, its variance is not defined. It is still possible to simply try to calculate the variance numerically from a finite time-series, but the resulting number is not the variance and grows with the sample size. When such a time-series is analysed in the following, it is purely for the sake of comparing the result of the application of the same algorithm that is also applied to the experimental time-series. Later in this thesis, simulated time-series will have pdfs that decay fast enough for the variance to be defined.

The combined time-series from single days as well as all days for each subject were analysed analogously. The complete analysis of all trials of all subjects and the combined time series of every day and every subject fills several hundred pages if all plots are shown. Therefore, only the combined time series of one subject for all days is shown in the next section. The range of the results found for other subjects is discussed. This is completely sufficient to compare the qualitative features of the experimental time series and the minimal model. More detailed results for the analysis of the of the other time series and for time series
3. Data analysis and comparison

Figure 3.2: Analysis of the combined time-series of the second series of trials of subject 1. A: Time series of the distances $Y = |y|$ versus iteration steps. B: Complementary cumulative distribution function of $Y$. C: Power spectrum. D: The variance of the cumulated magnitudes.

of different lengths can be found in Sec. 5.8. There, the variability in the experimental time series are be compared to the most advanced model presented in this thesis. Additional plots for other subjects can be found in Apdx. A.2

3.3. Results

Fig. 3.2 shows statistics of the combined time series of the trials of one subject. Similar results were obtained for all subjects. The experimental data show similarities as well as differences to the minimal model.

In particular, for all subjects, a power law is found in the probability densities. In individual trials, most tail exponents are between three and five. The onset of the power law scaling as well as the overall size of the fluctuations varies between different subjects.

The power spectrum is constant for low frequencies and then exhibits two regimes of power law decay which are more distinct for some trials than for others. For the first scaling regime, exponents are typically between 0.5 and 1.5. For high frequencies, scaling exponents between 2 and 3.5 are observed. The onset of the first scaling regime is found roughly at the order of 0.1 Hz.

At about 4 to 5 Hz, the transition to the second scaling regime occurs. Around 40 Hz, the spectra seem to level at a baseline. This is possibly due to quantisation noise. Because the second regime often exhibits one or more small peaks and levels levels for the highest frequencies, its overall appearance is less straight compared with the first regime.
3.3. Results

The scaling of the variance for the sums of values of $Y$ exhibits a Hurst exponent $H$ slightly below one for short times. This corresponds to positive correlations in the time series. A crossover to a Hurst exponent close to 0.5 begins where the sums are over times of the order of few seconds.

As we already have seen in the last chapter, the minimal model (Fig. 3.3) shows power law distributed fluctuations with a tail exponent of 2.0. We also know, that three subsequent values of $Y$ are correlated and that the process is otherwise uncorrelated. This is reflected by by a tiny dip at high frequencies in the otherwise constant power-spectrum. Notice the y-Axis in Fig. 3.3 C. The scaling in Fig. 3.3 D is consistent with uncorrelated random numbers, but restrictions mentioned in Sec. 3.2.3 D apply.

Errors

Suppose, that $\mu_0$ is the true scaling exponent underlying a finite data set. An advantage of the Hill estimator is, that the distribution of its reciprocal is known to be a Gaussian with

$$\left\langle \frac{1}{\mu} \right\rangle = \frac{1}{\mu_0} \quad \text{and} \quad \text{var} \left( \frac{1}{\mu} \right) = \frac{1}{r \mu_0^2}. \quad (3.7)$$

This has the consequence, that the distribution of $\mu$ is skewed, although its mean is unbiased. For small $r$, the most probable value of $\mu$ deviates from the mean and is therefore biased, but this bias is absolutely neglectable for the sizes of the time series analysed in this thesis.
Because the analysed data sets are quite large, the expected variance of $\mu$ is very small. As an example, take a simulation of the minimal model for $10^6$ iterations. Because this model shows few random bumps and a quick onset of the scaling regime in the pdf, the optimal cutoff is near the specified 70% limit (see above). Therefore, the first $1.6 \cdot 10^4$ ranks are used for the Hill estimator. 100 simulations of this model yielded a mean value of $\langle 1/\mu \rangle = 0.9975$ and a variance of $\text{var}(\mu) = 1.2 \cdot 10^{-4}$. The expected variance based on the number of ranks used in each estimation was only $0.6 \cdot 10^{-4}$. We see, that even for the minimal model and with a nearly constant cutoff rank the expected variance is lower than the observed one. There is also a slight bias compared to the analytic exponent which is exactly one. This is because the scaling behaviour only holds for the limit of infinitely large deviations. Small fluctuations are dominated by the additive Gaussian noise and the last remaining very small curvature disappears only slowly for increasing fluctuation sizes.

Similar results are obtained for different fixed cutoff ranks. The difference to the analytic scaling exponent decreases for smaller cutoff ranks, but the variance increases. The variable cutoff seems to increase the variance, but only slightly. Only for small cutoff ranks of the order of $10^3$ or even smaller, the bias disappears and the expected variance becomes as large as the observed one.

The experimental time series and some of the extended models including the continuous models show a much higher random variability and different shapes of the scaling regime onset. Hence, errors — even systematic ones — of the order of $10^{-3}$ are clearly below any heuristically expected uncertainty. Also, because of the size of the data sets and the high fluctuations in the tail, using only $10^3$ or less ranks clearly is expected to yield bad results. Furthermore, a cutoff to the scaling regime it is to be expected at some point because of the limited screen size. Therefore, it is appropriate to try to use ranks that are as close as possible to the onset of the scaling regime; even so this might cause small systematic deviations due to an incomplete convergence to the limit exponent.

Another possibility to measure the goodness of fit would be the Kolmogorov-Smirnov-Test. But this test again predicts very small errors for large data sets. For example, the hypothesis that the underlying distribution is the same for two simulations of the continuous model (Chap. 5) is repeatedly rejected by the KS-Test with the usual 5% significance level.

For the power spectrum, the amount of averaging and binning as well as the length and raggedness of the scaling regimes eliminates the usefulness of standard error estimates [1].

As a consequence of these findings, the variability found in the time series is investigated empirically in Sec. 5.8. There, experimental data and the continuous model with the most similar properties are compared. No errors are stated in the other analyses, but an uncertainty of at least several times the last digit specified should be expected.
4. Model extensions in discrete time

An application to realistic systems requires the models mechanism for generating power law distributed fluctuations to be robust with respect to the introduction of delays, memory and changes in the dynamics of the controlled system. Such extensions of the minimal discrete model are discussed in this chapter. These considerations will also provide insights important for the continuous time control considered in chapter 5.

4.1. Delays

Interaction delays are ubiquitous in motor control (e.g., \[25, 26, 2, 3\]). In the simplest case, one can introduce a delay \(n_e = 1, 2, 3, \ldots\) to shift the observations used for estimation further into the past. Then, Eq. (2.6) is replaced by

\[
\alpha_{t+n_e} = \frac{y_t}{y_{t-1}} + \alpha_t
\]

which will be referred to as system with estimator delay \(n_e\).

As mentioned above, \(n_e\) introduces a delay for the parameter estimator \(\alpha_t\), but not for the prediction \(\tilde{y}_{t+1}\) of \(y_{t+1}\). In the basic model \(\tilde{y}_{t+1}\) is always based on observing \(y_t\), even if \(\alpha_t\) is delayed by some \(n_e > 1\). Therefore, we always have a reaction delay \(n_r = 1\). Without such a delay, control would not be a problem at all, since the observed \(y_t\) could be used to remove it while still at time \(t\). Then, the controller could always set \(y_t\) exactly to zero without the need for any kind of estimations. Anyway, with a reaction delay of one, we still know that the system will evolve according to (2.1) before the next control action is performed.

Now consider reaction delays \(n_r > 1\). In this case, an action based on observing \(y_t\) is used to control \(y_{t+n_r}\). To compute the corresponding predicted state of the system \(\tilde{y}_{t+n_r}\), the controller has to take into account its own actions that will be performed in the meantime, but have already been planned. Fig. 4.1 shows the updated predictor. Eq. (2.3) is now written

\[
y_{t+1} = \alpha y_t - \tilde{y}_{t+1} + \beta_t.
\]

To predict the future behaviour of the system step by step starting at time \(t\), we have to introduce an intermediate estimate \(\tilde{y}_{t_t'}\). This estimation represents a hypothetical state of the system at time \(t + t'\), as predicted by the controller at a fixed time \(t\) based its then current knowledge.
4. Discrete-time extensions

This intermediate estimate will for \( t' \neq t \) not be used directly since the controller will remove \( \tilde{y}_{t+t'} = \tilde{y}_{t+|t+t'-n_r|} \) at time \( t + t' \). The concept is illustrated in Fig. 4.2. If we know \( y_t \), our estimation for \( y_{t+1} \) “before” the controller removes \( \tilde{y}_{t+1} \) is

\[
\tilde{y}_{t+1} = \alpha_{t+n_r} y_t
\]  

(4.3)

just as in the basic model. Note, that \( \alpha_{t+n_r} \) is now the parameter estimator that will be used for calculating \( \tilde{y}_{t+n_r} \), but is still based on observations that have been made at time \( t \) or before. If we have an estimate \( \tilde{y}_{t+t'} \) for the state of the system \( t' \) steps in the future without the control action (removing \( \tilde{y}_{t+t'} \)) that will be performed at time \( t + t' \), the intermediate estimation for \( y_{t+t'+1} \) is:

\[
\tilde{y}_{t+t'+1} = \alpha_{t+n_r} (\tilde{y}_{t+t'} - \tilde{y}_{t+t'}). \tag{4.4}
\]

One can now convince oneself, that starting at \( \tilde{y}_1 \), and subsequently subtracting the estimations \( \tilde{y} \) that we know will be used in each timestep and multiplying by our currently most recent parameter estimator \( \alpha_{t+n_r} \) yields

\[
\tilde{y}_{t+n_r} = \alpha_{t+n_r} y_t - \sum_{i=1}^{n_r-1} \alpha_{t+n_r} \tilde{y}_{t+i}
\]  

(4.5)

Reconsidering the estimator (2.6), we see that it can be obtained by simply rearranging the expectation value of (2.3). To take the correct control actions according to (4.5) into account, we have to modify the estimator to

\[
\alpha_{t+n_r} = \frac{y_t + \tilde{y}_t}{\tilde{y}_{t-1}}, \quad n_e \geq n_r. \tag{4.6}
\]

Eqns. (4.2), (4.5) and (4.6) represent the control system with reaction delay \( t_r \) and an optional additional parameter estimator delay \( n_r - n_e \). This system reduces to the minimal model for \( n_r = n_e = 1 \) (dropping the sum in (4.5)). Since a reaction delay introduces a fixed time scale into the system, it breaks the invariance against changes in the parameter \( \alpha \). This effect is discussed in more detail for the continuous model in Chap. 5.

Fig. 4.3 shows the power spectra for the model with both kinds of delays discussed above. While the delayed estimator (A) pronounces the low-pass filter also found in the minimal model, a reaction delay (B) causes a high frequency response that is more akin to a comb filter. The effects of a delayed estimator in combination with increased memory will be discussed at the end of this chapter in more detail. At this point, the reaction delay does not seem to help describing...
4.2. Optimal estimation and memory

A second extension of the basic model takes into account more past observations than only $y_t$ and $y_{t-1}$, i.e. a longer memory. For the basic model, we derived the control equation (2.6) by minimising the mean square error of the estimator. This is equivalent to maximizing the likelihood function of the predicted target position. It is straightforward to expand the second approach to an optimal estimator that uses more than two steps (see also [5]).

The dynamics of $y_{t+1}$ in each step in (4.2) are completely determined by $y_t$, $\tilde{y}_{t+1}$, $\alpha$ and $\beta_t$. This means, that the conditional probability density

$$P(y_{t+1}|y_t, \tilde{y}_{t+1}, \alpha, \sigma) = \mathcal{N}(\alpha y_t - \tilde{y}_{t+1}, \sigma^2)$$  \hspace{1cm} (4.7)

is a normal distribution whose mean is determined by the passive dynamics and the corresponding control action. See Fig. 4.4 for an illustration. If we assume this part of the equation to be given, there are no other dependencies on previous states of the system. The variance $\sigma$ of the conditional probability density is the same as the one of the independent and normal distributed $\beta_t$.

The conditional probability density for observing two subsequent values of $y$ given the preceding value, the corresponding predictor and, the systems parameters is then equal to the product of both variables densities:

$$P(y_t, y_{t-1}|y_{t-2}, \tilde{y}_t, \tilde{y}_{t-1}, \alpha, \sigma) = P(y_t|y_{t-1}, \tilde{y}_t, \alpha, \sigma) P(y_{t-1}|y_{t-2}, \tilde{y}_{t-1}, \alpha, \sigma).$$  \hspace{1cm} (4.8)

Hence, the likelihood function for the systems parameters based on $m$ previous observations

$$\mathcal{L}_m(\alpha, \sigma) = \prod_{i=0}^{m-2} P(y_{t-1}|y_{t-i}, \tilde{y}_{t-i}, \alpha, \sigma)$$  \hspace{1cm} (4.9)

Figure 4.4.: Power spectra for the models with A: an estimator that is delayed $n_e = 5$ steps. B: a reaction time $n_r = 5$ steps. $\alpha = 2$, $\sigma = .8$. The experimental data better. This changes in Chap. 5 when we consider a continuous controller for which a finite reaction time is inevitable. However, both kinds of delays lower the tail-exponent of the pdf. For long delays, $\delta$ gets close to (but lager than) one. For times shorter than the delay, hyperdiffusion occurs.
4. Discrete-time extensions

is the product of \( m - 2 \) univariate probability densities. According to (4.7), all factors are normal distributions. The explicit form of the likelihood function follows as

\[
L_m(\alpha, \sigma) = \prod_{i=0}^{m-2} \left( \frac{1}{\sigma \sqrt{2\pi}} \exp \left( -\frac{(y_{t-i} - \alpha y_{t-i-1} - \tilde{y}_{t-i})^2}{2\sigma^2} \right) \right)
\]

\[
= (2\pi \sigma^2)^{1-\frac{m}{2}} \exp \left( \frac{-1}{2\sigma^2} \sum_{i=0}^{m-2} (y_{t-i} - \alpha y_{t-i-1} - \tilde{y}_{t-i})^2 \right). \quad (4.10)
\]

The maximum likelihood estimator for \( \alpha \) given \( m \geq 2 \) observed values of \( y \) and the corresponding control actions is the value of \( \alpha \) that maximises \( L_m \)

\[
\alpha_{t+1} = \arg \max_{\alpha} L_m(\alpha, \sigma). \quad (4.11)
\]

Since maxima are unaffected by monotone transformations, we can also maximise the logarithm of this expression. To find the maximum, we demand the partial derivative of the log-likelihood function with respect to \( \alpha \) to be zero:

\[
0 = \frac{\partial}{\partial \alpha} \ln L_m(\alpha, \sigma)
= \frac{\partial}{\partial \alpha} \left( \ln (2\pi \sigma^2)^{1-\frac{m}{2}} - \frac{1}{2\sigma^2} \sum_{i=0}^{m-2} (y_{t-i} - \alpha y_{t-i-1} - \tilde{y}_{t-i})^2 \right)
= \sum_{i=0}^{m-2} (y_{t-i} - \alpha y_{t-i-1} - \tilde{y}_{t-i}) y_{t-i-1}. \quad (4.12)
\]

Solving for \( \alpha \) yields the estimator

\[
\alpha_{t+n_e} = \frac{\sum_{i=0}^{m-2} (y_{t-i} + \tilde{y}_{t-i}) y_{t-i-1}}{\sum_{i=0}^{m-2} y_{t-i-1}^2}. \quad (4.13)
\]

Since there is no other turning point in \( \alpha \) and the second derivative is strictly less than zero, this is the requested maximum likelihood estimator. This estimator has a memory for \( m \) past values of \( y \) and for its own actions that influenced the newer \( m - 1 \) of these values. The case \( m = 2 \) is equivalent to (4.6) and for \( n_e = 1 \) to the minimal estimator (2.6).

Numerical investigations indicate, that for the system with memory \( m \) (Eqns. (4.6) and (4.13)) and delays \( n_r = n_e = 1 \), the probability density \( p_m(y_t | \alpha) \) has a tail exponent \( \delta_m = m \), independently of the value of \( \alpha \) and the constant noise level \( \sigma \) (not shown).

Using (4.6) to replace \( \tilde{y}_{t-i} \), (4.13) can be written in the effective form

\[
\alpha_{t+n_e} = \alpha + \frac{\sum_{i=0}^{m-2} \beta_{t-i-1} y_{t-i-1}}{\sum_{i=0}^{m-2} y_{t-i-1}^2}. \quad (4.14)
\]
4.3. Exponential decaying memory

which resembles Eq. (2.7). Relative to the minimal model, the probability for large fluctuations is reduced because now the sum over \( m - 1 \) values of \( y^2 \) in the denominator has to be small to critically amplify \( \beta \).

4.3. Estimation with exponential decaying memory

For a more realistically shaped memory kernel, exponentially decaying weights \( e^{-i/\tau_m} \) with time constant \( \tau_m \) can be assigned to the summands in the numerator as well as in the denominator of (4.13) to replace the artificially box-shaped memory by an exponentially fading one. Taking the limes \( m \rightarrow \infty \), we can then write both sums as separate recurrence relations. This leads to a set of estimation equations with exponentially decaying memory

\[
A_t = (1 - \epsilon) A_{t-1} + (y_t + \tilde{y}_t) y_{t-1} \\
B_t = (1 - \epsilon) B_{t-1} + y^2_{t-1} \\
\alpha_{t+n} = \frac{A_t}{B_t} \tag{4.15}
\]

with \( (1 - \epsilon) = e^{-1/\tau_m} \). Together with (4.6), these equations represent a more plausible version of a closed loop controller for real systems. For \( \epsilon = 1 \), these equations are equivalent to the basic control system without memory and for \( \epsilon = 0 \), the memory becomes infinite. The characteristic decay time of the memory is \( \tau_m = -\ln{(1 - \epsilon)^{-1}} \), which becomes \( \tau_m \approx 1/\epsilon \) for small \( \epsilon \).

Note, that the above equation is equivalent to a low-pass filter where the input is amplified in such a way that the filters damping is compensated. Only the decay is affected by the filters time constant. This means, that for an infinite time-constant, the filter sums over all inputs.

The estimator (4.15) can also be written in a closed form that reflects the effective critical noise amplification

\[
\alpha_{t+n} = \alpha + \frac{\sum_{i=0}^{\infty} (1 - \epsilon)^i \beta_{t-i-1} y_{t-i-1}}{\sum_{i=0}^{\infty} (1 - \epsilon)^i y^2_{t-i-1}}. \tag{4.16}
\]

Finally, the control equation (4.15) could also be equipped with constraints to suppress unrealistically large estimates of \( \alpha \), for example, by adding a small positive term to \( B_t \). This extension of the system yields power law tails of \( p(y_t) \) which, however, then have an exponential cutoff. Because the inclusion of such a truncation complicates the systems dynamics without providing substantial benefits for the following discussion, it will not be considered here.

4.4. Properties of extended models

The tail exponent of the distribution of \( Y \) depends on the memory available to the controller and on delays. Larger memory leads to larger exponents, while
4. Discrete-time extensions

Figure 4.5.: Power spectra for the model with decaying memory with $\alpha = 2$; $\sigma = 1$. A: memory decay constant $1/\epsilon = 0.85$; estimator delay $n_e = 0$. B: $1/\epsilon = 0.7$ and $n_e = 10$.

Long delays reduce the exponents. The model with an exponential decaying memory allows for continuous adjustment of the pdf tail exponent. The model with $m$ steps of memory allows only for integer exponents. Otherwise, results are very similar. A controller with the minimal memory of two steps and positive delay leads to tail exponents smaller than 2. The power spectrum is always constant for low frequencies and shows a power law decay over one to two orders of magnitude at high frequencies. A model with longer memory but no delay shows a resonance peak in the power spectrum around 0.1 steps$^{-1}$ (Fig. 4.5 A).

Longer memory time constants broaden the peak and shift it to lower frequencies (not shown). Low-pass filtering caused by the addition of an estimator delay reduces the resonance. For longer delays, there may occur multiple resonance peaks that overlay the general low-pass filter. This happens, if the resonance is not sufficiently suppressed because the memory time constant is too long (Fig. 4.5 B). For a reaction delay, the characteristic frequency response remains when combined with memory (not shown).

Figure 4.6.: Model with decaying memory with $\alpha = 2$; $\sigma = 0.8$; $\epsilon = 0.85$ and $n_e = 10$. A: Time series. B: Complementary cumulative distribution function. C: Power spectrum. D: Variance of the cumulated magnitudes.
When memory and estimator delay are combined in such a way that resonance is suppressed two scaling regimes occur in the power spectrum. Nonstationarities like using a varying $\alpha$ throughout the simulation also suppresses resonances.

The scaling of the variance for the sums of values of $Y$ exhibits a Hurst exponent $H$ between one and $0.5$ for short times. However, $H$ is dependent on parameters and generally lower than observed experimentally. A crossover to a Hurst exponent close to $0.5$ begins where the sums are over times of the order of few seconds.

**Parameter dependencies**

Varying the model parameters makes it possible to cover the experimentally observed range of scaling exponents in the pdf and power-spectrum, while generating data which shares all properties of a given experimental time series at the same time turned out to be difficult. For example, it is problematic to reach higher exponents ($\geq 4$) in the pdf and suppress resonances without adding additional extensions like nonstationarities to the model. Fig. 4.6 shows a simulation where all exponents but $H$ lie within the experimentally observed range.

Simulations of the model show a high variability. Even for different simulations with the same parameters, the generated time series may have varying properties due to random fluctuations. As noted before, random variability is investigated in more detail in Sec. 5.8.

To get an impression of the parameter dependence of the tail exponent $\delta$ of the distribution of the distances $Y$, Fig. 4.7 shows $\delta$ versus memory length and delay. The exponent rises monotonically with memory length. Short delays decrease the exponent while for delays larger than 6, further increments have little effect.

**Non-stationarity and optimality**

To examine the influence of non-stationary circumstances, Fig. 4.8 shows the same representation as Fig. 4.7 but for the case of varying $\alpha$. In this situation, a long memory can be disadvantageous and reduces the exponent. To suppress large errors, the memory used by the controller has to be chosen according to the delay and the time scale of variation of the system.

![Figure 4.7: Tail exponent $\delta$ for different combinations of delay and decaying memory. Fitted using the Hill estimator, as described in Sec. 3.2. The rank-ordered absolute values of $Y$ have been averaged for 10 simulations with $10^9$ time steps each with $\alpha = 2$ and $\sigma = 0.8$.](image1)

![Figure 4.8: Tail exponent $\delta$ for simulations with parameter ranges as in Fig. 4.7 but with $\alpha$ chosen randomly from $\{2, 3, 4\}$ every 30 time steps.](image2)
4. Discrete-time extensions

to be controlled. A reduction of the exponent for a longer memory even occurs, when the rate at which $\alpha$ changes is very low compared to the delay length. The worst case for the controller is when $\alpha$ is constant over a time interval being exactly as long as the delay time. This situation yields exponents much below 2 that rise again, when $\alpha$ is changed in periods being shorter than the delay time (not shown).

In these simulations the controller was still able to control the system, because $\alpha$ was varied only over a limited range. This corresponds to the experiments, where the possible values for $\alpha$ were chosen in a range where the subjects where still able to stabilise the target. Large jumps in $\alpha$ can cause the system to become unstable. In different simulations the controller could handle much stronger variations in $\alpha$ when the intervals in which $\alpha$ stayed constant where longer than the delay, than in simulations, where these intervals were shorter than the delay.

Generally, the quality of the estimation of $\delta$ should be very good due to the large time series. Only for large exponents like 5 or higher, the convergence to the limit exponent for large fluctuations is very slow. Also, power laws with such high exponents cannot be reliably distinguished from other distributions in data sets of the size of the experimental ones.

The above findings raise the question, how the term optimal estimator is to be understood. The estimator with memory $m$ is optimal, when only $m$ observed timesteps are allowed to be used. The exponential decaying estimator is an ad-hoc extension of this optimal estimator. Optimal means optimal under the restriction of a given memory length and not globally. From Fig. 4.7 it becomes obvious, that it is globally optimal to use all previous observations if the systems properties are constant. Then, the longer the controller collects observations, the more information it obtains about $\alpha$.

In Fig. 4.8 we see that a limited memory can actually be optimal under certain circumstances. Because the exponent for the system without a reaction delay depends only on changes in $\alpha$, not its actual value. We further investigate this effect by having $\alpha$ perform a random walk. If this random walk is lowpass filtered, we can continuously change the time scale of a continuous variation in the system. This avoids the characteristic property of jumps, where the change is infinitely fast and happens only at singular time steps.

The dependency of the exponent on the memory decay constant and the characteristic time scale of change in $\alpha$ is shown in Fig. 4.9. Here, the memory decay factor $\epsilon$ (see (4.15)) is directly used on the memory axis. This allows to show the whole parameter space from the minimal model for $\epsilon = 1$ to a model with infinite memory for $\epsilon = 0$. Analogously, the lowpass filter’s time constant is char-

![Figure 4.9: Tail exponent $\delta$ for the model with no delay, decaying memory and a lowpass filtered random walk in $\alpha$ with mean 2 and drift 1. $\sigma = 1$. Fitted using the Hill estimator, as described in Sec. 3.2. The rank-ordered absolute values of Y have been averaged for 10 simulations with $10^9$ time steps.](image-url)
characterised by the decay factor $\epsilon_\alpha$. For $\epsilon_\alpha = 1$, $\alpha$ performs an uncorrelated random walk. For $\epsilon_\alpha = 0$, $\alpha$ is constant.

For most values of $\epsilon_\alpha$, very short memories with $\epsilon > 0.8$ yield the highest exponents. A long memory decay constant is only beneficial, if the time scale of variation in $\alpha$ is even longer. An infinite memory can even cause the system to become unstable because old observations that no longer carry any information about the current state of the system are never forgotten. A controller with an infinite memory is only stable, if $\alpha$ is constant. This case lies exactly on the right-hand boundary of Fig. 4.9. The slight increase of $\delta$ near the stability boundary is probably an artefact. For these simulations, the pdfs are very crooked and only marginally resemble the clear power law that usually characterises critical control models.

The impact of the variation in $\alpha$ on the exponent increases with an increasing drift, but it is independent of the mean of the random walk (not shown). If the Range of values which $\alpha$ can take is limited like in Fig. 4.8, an infinite memory can also be stable depending on the parameters. Then, the controller only incorporates the mean of alpha in its control. Depending on the parameters, this usually reduces the exponent drastically (not shown).

For different dynamics in $\alpha$, the landscape for $\delta$ may become very complex. Clearly, an optimal memory length always depends on the exact properties of the control system.

A reaction delay like in Eq. (4.5) or in the continuous model discussed in the next chapter introduces a fixed time scale to the system. This breaks the invariance against the $\alpha$ which determines the time constant of the uncontrolled dynamics. Then, $\alpha$ has to be restricted to a certain range of values. This causes complex interactions between the possible benefit of averaging over small changes in $\alpha$ and the dependence on its value and time scale of change. No clear picture arised from such investigations so far and therefore, they will not be discussed here any further.

Note, as a final remark, that it is not inevitable that the shortness of the memory is the consequence of an optimality principle. It could just as well be caused for example by biological restrictions.
5. Continuous critical control

A continuous controller is derived that is as similar as possible to the discrete-time ones considered before. The model properties will be compared to experimental data. Further, the shape of single peaks, signal to noise issues and the variability in simulated and experimental time series are investigated. Finally, the possibilities for the application of the model to systems with long-range correlations in time are discussed briefly.

5.1. Continuous dynamics without control

Consider a continuous dynamical variable \( y(t) \) that again denotes the deviation of a system from some target value. The stochastic differential equation (sde)

\[
\dot{y}(t) = \frac{1}{\tau} y(t) + \beta(t)
\]

(5.1)

defines a continuous control problem analogously to Eq. (2.1). Here, \( \tau \) is the time constant of the exponential growth. \( \beta(t) \) is a random variable with variance \( \sigma^2 \) and autocorrelation \( \langle \beta(t) \beta(t') \rangle = \delta(t-t') \), also called gaussian white noise. Note, that the unit of the variance is \( [\sigma^2] = \frac{1}{t} \). This sde is already a shorthand notation for a corresponding stochastic integral equation:

\[
y(t) = y(0) + \int_0^t \frac{1}{\tau} y(t') dt' + \sigma \int_0^t dW(t')
\]

(5.2)

The solution of the first integral is an exponential. The second one gives the Wiener Process \( W(t) \) with \( \sigma dW(t) = \beta(t) dt \). For a given \( y(t) \), we get for the distribution of \( y \) at time \( t + t' \)

\[
y(t + t') \sim \mathcal{N} \left( y(t) e^{t'/\tau}, t' \sigma^2 \right)
\]

(5.3)
as depicted in Fig. 5.1. Choosing \( t' = 1 \), we can relate the continuous equation to the time-discrete map (2.1) by identifying: \( \alpha = e^{1/\tau} \) and \( \beta_t = \sigma (W(t+1) - W(t)) \).

5.2. Pulsed control

To control Eq. (5.1), the first possibility is to remove an estimate \( \tilde{y}(t) \) of \( y(t) \) using delta-shaped pulses\(^1\) at certain times \( t_i \):

\[
\dot{y}(t) = \frac{1}{\tau} y(t) - \sum_i \delta(t - t_i) \tilde{y}(t_i) + \beta(t).
\]

(5.4)

\(^1\)Because \( \int_{-\infty}^{\infty} \delta(x) \, dx = 1 \) and \([dx] = [x] \), the unit of the delta distribution is \([\delta(x)] = [1/x] \).
Between two pulses, the system evolves like Eq. (5.1). If we know \( y(t_i) \) at time \( t_i \), we can directly state the solution for the expectation value of the system's state at time \( t_{i+1} = t_i + t_p \) and without the control pulse applied

\[
\langle y(t_i + t_p - 0) \rangle = y(t_i) \frac{e^{t_p/\tau}}{\sigma},
\]

(5.5)

where \( t_p \) is the time difference between two pulses. If we observe the system only at the times \( t_i \) where control pulses are applied, we can simply multiply the observed distance by some factor \( \alpha(t_i + t_p) \) to get an estimation for the distance when the next pulse is applied:

\[
y(t_i + t_p) = \alpha(t_i + t_p) y(t_i).
\]

(5.6)

The real parameter \( \alpha \) is unknown to the controller, but can be estimated analogously to the discrete case (4.6) using two recent pulses:

\[
\alpha(t_i + t_p) = \frac{y(t_i) + \tilde{y}(t_i)}{y(t_i - t_p)}.
\]

(5.7)

The time-series for the model with pulsed control is shown in Fig. 5.2. Extensions to other memory kernels can also be obtained like in Chap. 4. Control with reaction times \( t_r = n_r t_p \), \( n_r \in \mathbb{N}^+ \) can be achieved just as in Eq. (4.5) using

\[
\tilde{y}(t_i + t_r) = \alpha(t_i + t_r)^{n_r} y(t_i - j \cdot t_p) 
\]

(5.8)

Choosing \( t_p = 1 \) and observing the system only at the times \( t_i \) when control pulses are applied, we regain the discrete map. Between pulses, \( y(t) \) grows exponentially and changing the pulse interval \( t_p \) stretches the system in time. The pdf of the pulsed model has exactly the same shape as for the minimal model. However, the pulses cause small resonance peaks that are superimposed on the high frequency damping in the power spectrum. The scaling of the variance reveals hyperdiffusion for timescales of the order of \( t_p \) or below.
5.3. Continuous control

A continuous controller cannot remove the estimation $\hat{y}(t)$ from Eq. (5.1) completely at some point in time without reaching infinite velocities. Instead, it will continuously remove a term proportional to $\hat{y}(t)$. To stabilise the system, the constant of proportionality has to be bigger than $1/\tau$. Thus, we get

$$\dot{y}(t) = \frac{1}{\tau} y(t) - \gamma \dot{\vartheta}(t) \hat{y}(t) + \beta(t)$$ (5.9)

with the parameter estimator $\dot{\vartheta}(t)$ for $1/\tau$ that will also be used to compute $\hat{y}(t)$. $\gamma > 1$ is a gain factor that ensures, that the controller tries to pull $y$ faster to zero, than it will grow without any kind of control. For $\gamma = 1$, $y$ performs a random walk with zero mean, but no drift.

In the last chapter, we showed that control without delays pose no problem at all and how to deal with reaction delays when control actions are performed during the reaction time. In the continuous control case, there are no time steps or pulses. This means, that there will always be control actions performed during a delay. Applying the same approach as in Sec. 4.1 to Eq. (5.9), we again consider intermediate estimates $\hat{y}(t')|_{t-t_r}$ about the state of the system at time $t + t'$ based on the observations until time $t$. Precisely, we predict $y(t + t')$ without the control action at the same time. We start at the observed state $\hat{y}(0)|_{t-t_r} = y(t - t_r)$ and want to compute $\hat{y}(t) := \hat{y}(t_r)|_{t-t_r}$. To do this, we have to solve the equation

$$\hat{y}(t')|_{t-t_r} := \dot{\vartheta}(t) \hat{y}(t')|_{t-t_r} - \gamma \dot{\vartheta}(t) \hat{y}(t - t_r + t')$$ (5.10)

which is basically the same as Eq. (5.9), but with $1/\tau$ replaced by the estimator $\dot{\vartheta}(t)$ and without the noise which cannot be predicted anyway. The second term represents the true control movements that have already been planned and that will be performed during the reaction time. The control diagram is shown in Fig. 5.3. In comparison with the minimal model shown in Fig. 2.1, the forward model now includes the controllers own actions and a gain has been added.

Eq. (5.10) is a first order linear differential equation with inhomogeneity $\gamma \dot{\vartheta}(t - t_r + t') \hat{y}(t - t_r + t')$. The complete general solution is

$$\hat{y}(t')|_{t-t_r} = e^{\dot{\vartheta}(t)t'} \left( -\gamma \int_0^{t'} e^{-\dot{\vartheta}(t)t'} \dot{\vartheta}(t - t_r + t') \hat{y}(t - t_r + t') \, dt' + y(t - t_r) \right)$$ (5.11)
where $t' = 0$ denotes, that we don’t include the control action at time $t'$. Since the borders of the integral don’t contribute to its value this is actually unimportant, but makes it more obvious how to discretise the system for numerical simulations. For $t' = t_r$, we get the desired prediction

$$\hat{y}(t) = e^{\gamma(t)} \left( -\gamma \int_{0}^{t_r} e^{-\gamma(t')} \hat{y}(t - t_r + t') \, dt' + y(t - t_r) \right).$$

(5.12)

Moving the first exponential into the bracket and under the integral, one can see that Eq. (5.12) can be obtained from Eq. (5.6) by replacing the sum with an integral and scaling the control actions by $\gamma \hat{y}(t - t_r + t')$ like in Eq. (5.9).

In the minimal model, the optimal estimator can be obtained by simply rearranging the controlled dynamics (2.3), discarding $\beta$ and shifting $t$ by one time step leaving only known variables. We get a similar expression to estimate the parameter $1/\tau$ by analogously rearranging Eq. (5.9):

$$\hat{\theta}(t + t_r) = \frac{\hat{y}(t) + \gamma \hat{\theta}(t) \hat{y}(t)}{y(t)}.$$

(5.13)

In the discrete case, the estimator with exponentially decaying memory (4.6) can be obtained from Eq. (5.9) basically by expanding the fraction with the denominator and applying a lowpass filter with time constant $\tau_m$ to both numerator and denominator. To match the exact form, the input of the lowpass filter is amplified by the filters time constant. However, for the actual estimator this amplification numerically makes no difference after few iteration steps. The same procedure applied to Eq. (5.13) yields the parameter estimator with an exponentially decaying memory for the continuous controller:

$$\hat{A}(t + t_r) = -\frac{1}{\tau_m} A(t + t_r) + \left( \hat{y}(t) + \gamma \hat{\theta}(t) \hat{y}(t) \right) y(t)$$

$$\hat{B}(t + t_r) = -\frac{1}{\tau_m} B(t + t_r) + y(t)^2$$

$$\hat{\theta}(t + t_r) = \frac{\hat{A}(t + t_r)}{\hat{B}(t + t_r)}.$$  

(5.14)

Note, that $B$ can get arbitrarily small, but it will always be positive if $\gamma \neq 0$.

Again, the estimator can be written in a way that reveals its critical noise amplification like Eqs. (2.7), (4.14) and (4.16). Using (5.9), $\gamma \hat{\theta}(t) \hat{y}(t)$ in (5.14) is replaced. $A(t)$ and $B(t)$ are determined by inhomogenous first order linear differential equations. Therefore, they may be solved analogously to Eq. (5.10). This yields the effective form of the estimator

$$\hat{\theta}(t + t_r) = \frac{\int_{t_0}^{t} e^{r'/\tau_m} \left( \frac{1}{2} y(t') + \beta(t') \right) y(t') \, dt' + A(0)}{\int_{t_0}^{t} e^{r'/\tau_m} y(t')^2 \, dt' + B(0)}.$$

(5.15)

with the initial conditions $A(0)$ and $B(0)$. Since both integrals in the above equa-
tion grow with time, we can neglect the initial conditions and obtain
\[
\theta(t + t_r) \approx \frac{1}{\tau} + \frac{\int_{t_r}^{t} e^{t'/\tau_m} \beta(t') y(t') \, dt'}{\int_{t_r}^{t} e^{t'/\tau_m} y(t')^2 \, dt'} \quad \text{for large } t.
\] (5.16)

This effective estimator resembles the effective form of the minimal estimator as closely as possible in continuous time. It is also numerically a nearly perfect approximation to the exact estimator after iterating the system for a few times \(\tau\).

### 5.4. Discretising the continuous system

Applying the Euler-Maruyama method to Eq. (5.9), we get

\[
\dot{y}_{k+1} = \dot{y}_k + h \left( \frac{1}{\tau} \dot{y}_k - \gamma \theta_k \tilde{y}_k \right) + \sqrt{h} \beta_k
\] (5.17)

with \(y_k = y(t_k)\) for the discrete times \(t_k = b \cdot k, k \in \mathbb{N}^+\). The time interval from one timestep to the next one is \(h = t_r/n_r, n_r \in \mathbb{N}^+\) so there are \(n_r\) timesteps calculated during the reaction delay \(t_r\). \(\beta_k\) is a gaussian distributed random variable with variance \(\sigma^2\). Assuring that the same timesteps are used, we can now discretise Eq. (5.12), too:

\[
\tilde{y}_{k+n_r} = e^{\theta_{k+n_r} \cdot t_r} \left( -\gamma h \sum_{j=1}^{n_r-1} \left( e^{-\theta_{k+j} \cdot h} \theta_{k+j} \tilde{y}_{k+j} \right) + y_k \right).
\] (5.18)

Moving the first exponential into the bracket and under the sum, this equation is equivalent to Eq. (4.5) with \(t_p = h\) if we set \(\gamma \theta h = 1\). This discretisation is equivalent to a special generalisation of the time-discrete or pulsed controllers prediction of \(y\). Here, instead of trying to bring \(y\) to zero in every time step, it is just pushed towards zero allegedly faster than it will grow without control.

The discretised version of the continuous parameter estimator with exponentially decaying memory (5.14)

\[
\begin{align*}
A_{k+n_r} &= (1 - b/\tau_m)A_{k+n_r-1} + (y_{k+1} - y_k + h \gamma \theta_k \tilde{y}_k) y_k \\
B_{k+n_r} &= (1 - b/\tau_m)B_{k+n_r-1} + b y_k^2 \\
\theta_{k+n_r} &= \frac{A_{k+n_r}}{B_{k+n_r}}
\end{align*}
\] (5.19)

bears some similarity to the discrete-time version (4.15), too. For \(\tau_m = h\), we obtain the estimator without memory. In difference to the discrete map, here the shortest possible memory only depends on the increment of \(y\) at a given time step and not actually on two timesteps. The exponent in this case is numerically nearly one. This means, that all values for \(y\) have nearly the same probability!
5. Continuous critical control

Sometimes, simulations will fail because \( \gamma \) exceeds the the largest machine size number. Because of this technical limitation, we will only obtain time series with an exponent that is slightly bigger than one and with a limited number of time steps. These problems do not occur for \( \tau_m > h \), or if a truncation is introduced.

As we have seen before, the pulsed control system is equivalent to the discrete time model if only the times at which control pulses are applied are considered. Between pulses, the system evolves according to the exact solution of the uncontrolled dynamics. This is the difference to the discretised continuous system where the uncontrolled part of the dynamics between iteration steps are linearised.

5.5. Properties of the continuous model

In contrast to the time-discrete model, we can choose realistic values for the time constants and the reaction delay. To match the experimental recording frequency of 85 Hz, the iteration intervall for the discretised model (5.17), (5.18) and (5.19) is set to \( h = 11.8 \text{ ms} \). Fig. 5.4 shows an analysis of the continuous model for realistic parameters. This means, that \( \tau = 1/4 \text{ s} \) like in Fig. 3.2 and that the other parameters have been adjusted to yield results that are similar to the ones for the experimental data sets.

The continuous model shows power law distributions in the pdf. Like in the discrete case, delays reduce the tail exponent \( \delta \) while a longer memory increases it. There is also an interaction with the the gain. High values of \( \gamma \) lead to larger fluctuations if an inaccurate estimation is used. This causes a larger scaling of the fluctuations. If the other parameters are set as in Fig. 3.2 gains above three lower the tail exponent. For high gains around 10, the system cannot be controlled. The dependence of the tail exponent on memory and gain and the dependance on nonstationarities is discussed below.

The power spectrum is constant for low frequencies and then exhibits two regimes of power law decay. The exponent of the first regime is dependent on \( \gamma \). Lower gains lead to lower scaling exponents. For values of \( \gamma \) significantly larger than two, the difference between the two scaling regimes is too small to be observed and a resonance peak may occour.

The first exponent also depends on the memory time constant \( \tau_m \) used by the estimator. Increasing the memory length also ingreases the exponent. The first scaling exponent can even exceed the second one. For long time constants, a resonance peak may occur even for low gain factors.

The second scaling exponent does not depend on the model parameters. Its value is roughly 2.5. Like in the experimental data sets, the second regime exhibits one or more small peaks and levels near the Nyquist frequency. The shape of the onset of the second regime and the distribution of the small peaks is not completely random. Partially, it is determined by the model parameters. Therefore, the fitted exponent and the appearance of the regime may be weakly influ-
5.5. Properties of the continuous model

![Figure 5.4](image)

Figure 5.4: Analysis for the continuous model with:
- $\tau = 250 \text{ ms}$;
- $\sigma = 2.5 \text{ s}^{0.5}$;
- $t_r = 176 \text{ ms}$;
- $\gamma = 1.1$;
- $\tau_m = 130 \text{ ms}$.


enced by the parameter choice. However, considering random fluctuations and different possible cutoffs used for fitting, the exponent usually is between two and three.

The frequency at which the transition to the second scaling regime occurs depends on the reaction time of the controller. Onset frequencies around 4 to 5 Hz like observed experimentally can be observed for reaction times between 170 ms and 200 ms. This is consistent with the range of simple reaction times for expected events. Note, that the time scales corresponding to the observed change in the behaviour in the frequency domain are in the range of 200 to 250 ms. This is the same range that has been reported for reaction times in tracking tasks (see Sec. 1.2).

The scaling of the variance for the sums of values of $Y$ exhibits a Hurst exponent $H$ slightly below one for short times. For all "realistic" parameter combinations, this exponent lies between 0.9 and 1.0. A crossover to a Hurst exponent close to 0.5 begins where the sums are over times of the order of few seconds.

**Parameter dependence of the tail exponent**

The parameter dependence of the tail exponent $\delta$ of the continuous model has been investigated analogously to the extended discrete model in the last chapter. Here, the memory decay constant $\tau$ and the gain factor $\gamma$ have been varied. Other parameters had presumably realistic values as in Fig. 5.4. Results are shown
5. Continuous critical control

in Fig. 5.5 \( \delta \) increases for longer memories. For short memory lengths, a short gain between one and two maximises \( \delta \). Above memory decay constants of approximately 150 ms, \( \delta \) decreases monotonically with \( \gamma \). For a slowly decaying memory and a low gain, the pdf can get distinguishable from a Gaussian.

Fig. 5.6 shows the same model for the same parameter combinations, but with \( \tau \) switched every second just like in some of the experiments. In contrast to the discrete time model, this switching has almost no effect on the controllers performance. This is probably due to the smoother continuous control strategy that averages over small changes in \( \tau \).

As for the discrete model with a reaction delay, the tail exponent of the continuous model is independent of the time constant \( \tau \). For small values of \( \tau \) the system becomes unstable. Fig. 5.7A and B show the time series of the model just before it becomes unstable. The dynamics is dominated by oscillations. Because \( y \) is larger than the noise floor for most of the time, large fluctuations become very unlikely. Further decreasing \( \tau \) causes the oscillations to grow exponentially and the system is unstable (C).

Fig. 5.7D shows the how \( \delta \) grows quickly when \( \tau \) is decreased until the system suddenly becomes unstable. For time constants larger than 1/4 s, changes in \( \tau \) cause only small changes in \( \delta \).

5.6. Scaling the noise level

5.6.1. The shape and distribution of peaks

The continuous-time model allows to use real units for the time-constants and for simulations with the same step length as the experimental data sets. Hence, it is possible to compare the structure of the time-series on short time scales. While the peaks in the discrete-time model have no inner structure, the sampling frequency in the experimental data-sets is high enough to resolve the structure of single peaks. Fig. 5.8 at a small part of the time-series also shown in Fig. 5.4. The large peak is surrounded by small fluctuations, raises suddenly and decays exponentially.

The time-series shown in Fig. 5.9 is a part of the experimental data shown in Fig. 3.2. Here, the single Peaks have a more ragged structure without a characteristic decay. Instead, they occur in short bursts lasting up to a few seconds. What is the reason for
5.6. Scaling the noise level

Figure 5.7.: continuous model with $t_r = 176$ ms; $\gamma = 1.1$; $\tau_m = 130$ ms; $\sigma = 2.5$ s$^{-0.5}$. Time series for A, B: $\tau = 100$ ms and C: $\tau = 83$ ms. D: Tail exponent for different time constants of the uncontrolled growth. Error bars indicate standard errors obtained from 20 simulations each.

Figure 5.8.: a small part of the time series of the continuous model also shown in Fig. 5.4

Figure 5.9.: a small part of the experimental time series also shown in Fig. 3.2

this qualitatively different behaviour? Recall the effective form of the estimator for the basic model (2.7) and the continuous one (4.16). The term that causes random deviations from the correct estimation is large for small values of $y$ in the denominator vice versa. For small $y$, the estimator fits the noise and may cause huge errors in the control actions. Once the damage is done, the values of $y$ are
very large compared to the constant additive noise floor and the real dynamics can be estimated with high precision. Now the controller pulls \( y \) back to zero with a time constant that is determined by the gain factor \( \gamma \). Only after \( y \) has reached the noise floor and stayed there long enough for the true dynamics to fade away in the estimators restricted memory, the procedure may start all over again. Visual inspection of the experimental time-series depicted in Fig. 5.3 shows, that the real system behaves somewhat differently. Obviously, the system is still disturbed by some noise even for large values of \( y \). The assumption that there is no limit on the precision of the controller if \( y \) is large enough compared to the driving noise is of course somewhat unrealistic.

5.6.2. Fitts' Law

The most well-known constraint on the precision of movements is Fitts' Law [35]. This speed-accuracy tradeoff formulated by Fitts in 1954 had an exceptional influence on fundamental and applied research on human movement. Knowledge of Fitts' Law is not essential to understand the model extension that will be made below and it is not clear if both effects have a common cause. Yet, they are at least closely enough related to make a short discussion seem worthwhile.

When reaching for a target, the movement time \( T \) required to reach the target depends on its width \( W \) and the amplitude \( A \) of the movement as

\[
T = a + b \log_2 \left( \frac{2A}{W} \right) \tag{5.20}
\]

with proportionality constants \( a \) and \( b \). The basic concept that remains unchallenged and underlies all variations of Fitts’ Law that have emerged in the past 50 years is that of an index of difficulty \( I_d = f(A/W) \) [37]. Originally, \( I_d \) was meant as a measure of movement information and therefore was measured in bits (hence the usage of \( \log_2 \)). Fitts’ idea was, that the combined visual- and motor system has a performance capacity \( I_p = I_d/T \) measured in bits/time that is constant over a range of task conditions. Eq. (5.20) is obtained in analogy to the channel capacity theorem by Shannon and Weaver [36]. \( A/W \) is then the signal to noise ratio. Today, \( I_p \) is usually treated as an ordinary dimensionless number, no longer referencing to the information concept.

Fitts also developed a new protocol where subjects had to alternately tap two rectangular plates of width \( W \) that were separated by a distance \( A \) as fast as possible. Missing the targets was penalised with a waiting time. This task was called the continuous task, because the subjects had to continuously move from one target to another. Other tasks including transferring discs between pins or pins between holes were performed, too. These tasks were called discrete tasks. Values of \( I_p \) for optimal performance were found to vary between 10-12 bits/s. A constant index of performance has not only been confirmed for the tasks in Fitts’ original experiments, but also for numerous other tasks. Fitts’ Law has proven especially valuable in human-computer interaction.
Unfortunately, Fitts’ Law could be modeled in many different ways. Models stem from fields as diverse as dynamics system theory, noise properties of neuronal firing and computational constraints in movement planning. Thus, the constraints it provides seem to be too unspecific to yield clear insights into its cause. Fitts’ Law has also been shown to be consistent with the closed-loop step-response of a delayed first order controller in the complete absence of noise. In other words, the time an exponential decay needs to approach the target value up to the allowed deviation is described by (5.20). As recently pointed out by [30], this can only be achieved by a predictive controller. When the delay is not moved out of the feedback loop by state prediction, the delay cannot be separated from the transfer function and the system’s step response is not consistent with Fitts’ Law.

The last point highlights the following problem: even in the so-called continuous task considered by Fitts, only the deviation from the goal (hitting the target) has been measured and not the fluctuations in the trajectory. In the balancing case, just these fluctuations that occur far away from the goal ($y = 0$) have to be explained. While intuitively it seems like the constrained performance in pointing tasks reliably described Fitts’ Law and the missing constraint in the balancing model could be related, it is not clear how to transfer the concept to the fluctuations considered here. Also, since the model shows an exponential decay after large fluctuations, it may already be considered consistent with Fitts’ Law if we only take into account the time to reach the target and not the idea of constant performance. On the other hand, this decay is only a first order approximation anyway since experimental velocity profiles in pointing tasks are bell shaped and the trajectories show perturbations by signal-dependent noise.

5.6.3. Limiting the estimators maximum precision

Even though it is not clear if the observed perturbations are connected to Fitts’ Law it is appropriate to mention, that the following approach is loosely related to Fitts’ original idea. Intuitively, it is clear that any process that involves the transmission of information will be limited by the considered system’s maximum signal-to-noise ratio. In its previously considered form, the estimator can achieve arbitrary precision if $y$ is large enough.

In the effective minimal estimator (2.7), we can impose a limitation to the information that can be obtained about the parameter $\alpha$ by adding an additional random variable $\beta'$ with variance $\sigma'^2$ to it:

$$\alpha_{t+1} = \alpha + \frac{\beta_{t-1}}{y_{t-1}} + \beta'_{t+1}. \quad (5.21)$$

Inserting into Eq. (2.3) which describes the controlled system’s dynamics yields

$$y_{t+1} = y_{t} \left(\frac{\beta_{t-1}}{y_{t-1}} + \beta'_{t}\right) + \beta_{t}. \quad (5.22)$$
5. Continuous critical control

We can now combine both random variables at time $t$ into a single new term for the driving noise $\sigma''_t \beta''_t = y_t \beta'_t + \beta_t$. $\beta''_t$ is a normal distributed random variable. The variance of the noise driving the process $\sigma''^2 = 1/2 (\sigma^2 + \sigma^2 y_t^2)$ is now scaled by a function with a constant part and a dependence on the observed value of $y_t$ that is used for predicting $y_{t+1}$. In the following, we will drop the two primes from the new random variable for convenience. The new form of the effective dynamics is then

$$y_{t+1} = \frac{\beta_{t-1} y_t}{y_{t-1}} + \sigma_t \beta_t. \quad (5.23)$$

The only difference to the minimal model $[2.8]$ is, that now the scaling of the noise is not constant. Note, that there is no distinction between a limitation to the knowledge about $\alpha$ that can be obtained and a limitation to the precision with which a movement based on this knowledge can be performed. This model already shows some clustering of peaks into short bursts. Increasing the scaling component of $\sigma_t$ reduces the tail exponent and can even make the model unstable. But what we are really interested in, is an extension to the continuous model.

When adding noise to the estimator $\theta$ in the continuous model $[5.14]$, it is not immediately clear how this is equivalent to scaling the noise with $y$ since $\theta$ is also used in the predictor $\hat{y}$ $[5.12]$. However, numerically this has the same effect as changing the dynamics of the continuous system to

$$\dot{y}(t) = \frac{1}{\tau} y(t) - \gamma \theta(t) \hat{y}(t) + \sigma(t) \beta(t) \quad (5.24)$$

where

$$\sigma(t) = \sigma + \sigma_y |y(t - n_r)|.$$  

Here, we will prefer this form over explicitly adding noise to the estimator because it only includes one random variable. Since the noise is assumed to be inherent in the controlling subjects’ CNS anyway, this should just be taken as an approximation to the effect, that here is some dependency of this noise on the last observed value of $y$. Furthermore, it is not essential for the formation of clusters that the value of $y$ used for scaling is delayed. If the delay is reduced or removed, the impact of scaling the noise on the dynamics is even enhanced.

Fig. 5.10 shows the continuous model with scaling noise defined by Eqns. $[5.24]$, $[5.12]$ and $[5.14]$. Now, the shape and distribution of the peaks qualitatively matches the experimental time-series. Fig. 5.12 shows the dynamics of the estimator $\theta$ for the model with constantly scaled noise. The large errors of the controller are followed by very good estimations of the real parameter. In Fig. 5.11 scaling the noise with the observed distance effectively introduces a minimal noise level in the estimator. Also, there seem to be less extreme deviations in the estimator

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Note, that without a delay $\partial \sigma(t, y)/\partial y \neq 0$. This leads to differences whether the process is interpreted using the Itô or the Stratonovich stochastic calculus.
5.6. Scaling the noise level

since small, but non-vanishing errors are sufficient to delay requiring the target during bursts.

Eq. (5.20) may lead to the assumption, that the noise should actually be scaled by the velocity of the controllers movement. That is, by the control action $\gamma \dot{\theta}(t) \tilde{y}(t)$. This is also equivalent to a noisy gain where $\gamma$ performs random fluctuations. Such an extension does lead to stronger fluctuations for high values of $y$ compared to the model without scaling the noise, but these are only fluctuations around an exponential decay. Bursts of large fluctuations like the ones observed in the experimental time cannot be obtained with this approach.
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**Figure 5.13:** Analysis of the same experimental data also as in Fig. 3.2. Shown for comparison with Fig. 5.14. A: Time series of the distances $Y = |y|$ versus iteration steps. B: Complementary cumulative distribution function of $Y$. C: Power spectrum. D: The variance of the cumulated magnitudes.

### 5.7. Properties of the Continuous Model with scaling noise

The extension of the continuous model to include a noise term that is partially scaled with the amplitude of the fluctuations can generate large fluctuations in short bursts like the ones observed in the real time series. The next question is of course, how this effect influences the statistical properties of the model. As it turns out, the variable noise scaling factor $\sigma(t)$ has some influence on the tail exponent $\delta$ of the pdf. As the proportionality constant $\sigma_y$ gets close to one, $\delta$ drops quickly. For $\sigma_y > 1$, the model becomes unstable. Increasing $\sigma_y$ also lowers the exponent in the first scaling regime in the power spectrum and shifts its onset to lower frequencies. The constant part $\sigma$ of $\sigma(t)$ still determines the overall scale of the fluctuations but doesn’t influence the other properties of the system.

Now, that the model can reproduce so many features of the real time series, it is quite tempting to try to create an artificial time series that looks as similar to
5.7. Properties of the Continuous Model with scaling noise

Figure 5.14: Results for the continuous model with scaling noise with $\tau = 250$ ms; $\sigma = 2.5$ $s^{-0.5}$; $\sigma_y = 0.8$ $s^{-0.5}$; $t_r = 176$ ms; $\gamma = 1.2$; $\tau_m = 190$ ms. A: Time series of the distances $Y = |y|$ versus iteration steps. B: Complementary cumulative distribution function of $Y$. C: Power spectrum. D: The variance of the cumulated magnitudes.

A real one, as possible. Fig. 5.14 shows the statistical analysis for the simulation also shown in Fig. 5.10. The parameters have already been adjusted to match the properties of the experimental time series that has been investigated before. It is shown again in Fig. 5.13 for convenience. Also, both time series now have the same length. Both time series share really similar properties. The main difference seems to be, that the onset of the power law in the pdf of the simulation is a bit smoother, but maybe even this could be further optimised by tweaking $\sigma$ and $\sigma_y$. However, this comparison should not be taken too seriously anyway. Given the models conceptual simplicity, it is rather surprising, that it can reproduce so many features of the experimental time series at all. What actually is of interest is to compare the variability of the model properties with an experimental time series that has very similar properties. Before this question is discussed there are two aspects left to complete the comparison with the constant noise model.
5. Continuous critical control

![Figure 5.15: Tail exponent δ for the continuous model with scaling noise for different combinations of gain and decaying memory. Fitted using the Hill estimator, as described in Sec. 3.2. The rank-ordered absolute values of Y have been averaged for 10 simulations with $2 \cdot 10^8$ time steps each with $\tau = 250$ ms; $\sigma = 2.5$ s$^{-0.5}$; $\sigma_y = 0.8$ s$^{-0.5}$; $t_r = 176$ ms. Stripes mark instability.](image)

Figure 5.15 shows the dependence of the tail exponent for the continuous model with scaling noise defined by Eqns. 5.24, 5.12 and 5.14. The inclusion of a scaling noise level causes changes in comparison to Fig. 5.5. $\delta$ still increases monotonously with $\tau_m$, but it does so much slower than for the constant noise model. The increment in $\delta$ decreases with increasing memory lengths. Above the shown parameter range, $\delta$ is nearly independent of $\tau_m$. $\delta$ shows a maximum for values of $\gamma$ between one and two. The maximum becomes more pronounced, shifted towards $\gamma = 2$ and tops around $\delta = 7$ for long memory time constants. The steep increase in $\delta$ near the left and bottom boundaries are just artefacts. There, the exponent is actually just slightly above one. For such low exponents, a distortion of the power law scaling appears in the rank ordered events that has nothing to do with a change of regime [1]. This effect fools the algorithm for determining the cutoff. When the cutoff is determined by the minimum of the slope of $\delta$ when varying the cutoff (see Sec. 3.2.2), the exponents near one are fitted correctly. However, this method has a high variance when fitting higher exponents.

The dependence of $\delta$ on $\tau$ for the model with scaling noise is quite different compared to the model with constant noise discussed before. The time series just before the model becomes unstable shown in Fig. 5.16 A and B now exhibits a high degree of irregularity and very large clusters of fluctuations. This is due to the effect, that larger values of $\gamma$ not immediately lead to good corrections any more. Increasing $\tau$ lets the short bursts grow into large clusters of fluctuations until the size of the clusters is not constant over the time series any more, but grows exponentially C. This usually causes the very last cluster in any part of the time series to be orders of magnitudes larger than any preceding fluctuations in the time series. The dependency of Fig. 5.16 D close to instability is now reversed in comparison with the constant noise case. For small values of $\tau$, the exponent decreases because the size of the clusters grows.

5.8. Variability of the simulated and experimental time series

What is the amount of randomness in different simulations of the model with the same parameters and how does this compare with the experimental data? Let’s assume, that fundamental characteristics of the subjects control strategy are captured by the model. Then, this comparison may give an impression of the least amount of variability to be expected in real time series even if the controlling subject does not change any properties of its control strategy.
Before, we analysed the combined time series of all trials of all days from a series of balancing experiments of one subject (Fig. 5.13). Tab. 5.1 shows the results for all subjects and for the simulation shown in Fig. 5.14. The data also shown in Fig. 5.13 is labelled “subject 1b”.

To investigate the amount of randomness in these results, the variability of subsets of the data for each subject is compared. Tab. 5.2 shows the means and standard deviations of the results when the combined time series of each day are analysed separately. Also, 20 simulations with identical parameters have been analysed. Tab. 5.3 shows the same analysis, but for each trial analysed separately. Here, the simulations have been shortened to the length of one trial. Standard errors for the means can be obtained by dividing the standard deviations by the square root of the number of time series analysed separately.

Relative standard deviations of several percent are found for the obtained exponents. On the other hand, the randomness observed when the analysis methods are kept constant are smaller than the assumable uncertainty. This is emphasised by the difference in the means of the same exponents between the three tables. Especially for the power spectra, the relative uncertainties caused by sensitivity to the cutoffs can be up to 50% although it has been tried to fit the same parts of the spectra in all three cases. The reason is that two cutoffs are necessary and the scaling regimes span less than 1.5 orders of magnitude on the frequency axis. There are also slight differences to the numbers published in [8]. This is because there the power spectra were less averaged and not binned and therefore had a higher amount of noise which made determining the cutoffs even more difficult. Also, a few trials have been discarded. These were slightly too short for an un-
5. Continuous critical control

| Simulation | 3.9 ± 0.3 | -1.07 ± 0.09 | -2.53 ± 0.08 | 1.954 ± 0.005 |
| Subject 1b (4) | 4.0 ± 0.3 | -0.9 ± 0.2 | -2.6 ± 0.1 | 1.931 ± 0.008 |
| Subject 1a (7) | 4.4 ± 0.7 | -1.0 ± 0.3 | -2.4 ± 0.2 | 1.94 ± 0.02 |
| Subject 2 (4) | 3.3 ± 0.3 | -0.8 ± 0.3 | -3.5 ± 0.2 | 1.961 ± 0.007 |
| Subject 3 (4) | 3.6 ± 0.2 | -0.7 ± 0.1 | -3.1 ± 0.3 | 1.960 ± 0.006 |
| Subject 4 (5) | 3.2 ± 0.4 | -1.1 ± 0.2 | -3.3 ± 0.2 | 1.947 ± 0.005 |
| Subject 5 (4) | 3.5 ± 0.2 | -0.9 ± 0.1 | -3.4 ± 0.2 | 1.951 ± 0.007 |
| Subject 6 (5) | 4.1 ± 0.2 | -0.76 ± 0.05 | -3.49 ± 0.08 | 1.942 ± 0.005 |
| Subject 7 (4) | 4.3 ± 0.2 | -0.5 ± 0.2 | -2.95 ± 0.09 | 1.930 ± 0.003 |

Table 5.1: Statistical properties of the simulation shown in Fig. 5.14 and the combined experimental time series for each subject. The corresponding plots for subject 1b are shown in Fig. 5.13.

Table 5.2: Properties of the same data as in Tab. 5.1 but with the combined time series for each day of each subject analysed. For each subject, the results are stated in the form: mean ± standard deviation. For the simulation, 20 runs with identical parameters were used.

Table 5.3: The same analysis as in Tab. 5.2 but with each trial analysed separately. The first 14950 events of each of the same simulations were used. This equals the number of events in each trial.

The numbers in brackets denote the number of time series that have been analysed separately. The employed methods are described in Sec 3.2. For the power spectrum, it turned out, that the transition between the two scaling regimes occurs nearly at the same frequency for all trials of the same subject. Therefore, the position and width of the transition has been estimated by visual investigation of the power spectra for each subject. For the automated analysis, 20 bins before and after this transition were used as cutoffs.

Subject one performed two series of recording balancing data on subsequent days. Between both series, there was a pause of several weeks. Uncontrolled growth of \( Y \) had a time constant of \( \tau = 1/3 \) s for the first series (a) and 1/4 s for the second one (b). For subject two and three, \( \tau = 1/3 \) s. For subject 4 – 7, 1/\( \tau \) was switched every second to a value in \( [3, 4, 5, 6] \). The parameters in the simulation have been adjusted to match the properties of the second series of subject one.
5.8. Variability of the simulated and experimental time series

**Figure 5.17.**: Complementary cumulative distribution functions for A: the combined time series for each day of the same experimental data as shown in Fig. 5.13 and C: the simulation shown in Fig. 5.14 together with 19 additional runs with identical parameters. Power spectra for the same B: experimental and D: simulated time series.

**Figure 5.18.**: The same data as in Fig. 5.17 but for each trial analysed separately and only the first 14950 events of each of the same simulations. Complementary cumulative distribution functions for A: the experimental trials and C: the simulations. Power spectra for the same B: experimental and D: simulated time series.
known reason which hadn’t been discovered before. However, these differences are smaller than the amount of uncertainty that has to be assumed anyway.

The one exception from the above considerations is the Hurst exponent. Here, values are constantly just slightly below one with relative uncertainties of less than one percent.

The amount of randomness grows for shorter time series. The standard deviation for the simulations is within the lower range of that of the experimental data. Surprisingly, the differences between the first three scaling exponents for the simulations and the data used for comparison are even less than two standard errors for the repeated simulations. The exception is the Hurst exponent, but the relative difference is only about one percent and the simulated exponent still lies within the small range of values observed for other subjects.

Plots for the cumulative densities and the power spectra for the time series for each day and the single trials for the same data as in Fig. 5.13 are shown in Fig. 5.17 and Fig. 5.18. Additional plots can be found in Apdx. A.2.

From the above results, a few conclusions can be drawn despite the high uncertainties. Pdf Tail exponents are significantly different between subjects and range from about three to 4.5. The first exponent in the power spectra is between 0.5 and 1.7 and the second one between 2.4 and 3.7. Again, there are significant differences between different subjects. Significantly here means, that the tendency of having low, high or average values for each particular exponent of each subject when compared to the other subjects is consistent across complete, daily and single-trial data. No characteristic property is found, that differentiates the subjects with constant time constants $\tau$ of the instability from the ones, where $\tau$ is switched.

The slower onset of the scaling regime in the pdf of the simulations compared to the experimental data mentioned earlier turns out to be persistent across different time series. However, this is not a general characteristic of the model and changes for different parameter combinations. Also, note that there is no quantisation noise in the model. In the experiments, the components of $Y$ are measured in discrete pixels and the mouse has limited sensitivity. This is expected to have the largest effect on the smaller fluctuations. It has also been confirmed, that the power law tail is not truncated in larger simulations (not shown).

The qualitative shape of the spectrum seems to be characteristic for each subject with the most amount of variability found in the offset of the spectral energy. While the first scaling regime is still relatively straight, the second one shows small superimposed resonances that seem to be characteristic for each subject and for certain parameter sets in the model. Also, the position of the transition between the two scaling regimes varies little between different trials of the same subject. These effects have not been reported before and are only visible due to the more aggressive noise reduction in this analysis.

The second scaling regime is for some subjects characterised by exponents that are systematically higher than three. This cannot be achieved by the model, because the second regime represents frequencies outside of the controllers active
response. This suggests, that the observed exponents are caused by passive filtering characteristics of the system, that are not modelled. This hypothesis is supported by simulations where the model was driven by lowpass filtered noise. For filter time constants of about one second, scaling in the power spectrum with an exponent above three can be obtained (not shown).

5.9. Correlations in time, volatility and other systems

As we have seen, the effective level of the noise driving the system is not constant. Adding a term proportional to $|\gamma|$ to $\sigma$ as in Eq. (5.24) already causes the fluctuations to occur in short bursts or clusters like in the experimental balancing data sets. Although further extensions to the model are not necessary as long as no additional experimental effects have to be explained, it should be noted that there are many other possibilities for the dynamics of the noise level. The motivation may not only be to incorporate possible new findings regarding the dependence of the noise level on the balancing dynamics in the future. Since balancing is far from being the only field where critical fluctuations are observed, it may be desirable to try to apply the formalism of critical control to other systems, too. Therefore, we will broaden the focus in this section slightly to give a perspective on modelling possibilities through different dynamics in $\sigma$.

For instance, financial markets can be considered control systems that optimise the dynamics of prices [6]. The fluctuations observed in price changes and in balancing experiments actually have quite a few similarities. However, one of the most important “stylised facts” [31] about financial time series are the so-called volatility clusters. The volatility is typically considered to be the standard deviation of the fluctuations in price changes within a specific time horizon. In diffusion models for the price, the volatility is the scaling factor $\sigma$ of the driving noise. While the scale of the fluctuations in virtual balancing varies on a time scale only slightly slower than the control dynamics, the volatility in markets varies much slower. Actually, slow dynamics in the scaling of stochastic fluctuations can be observed in many other systems like turbulent fluids, too.

The volatility causes higher order correlations between values of the observed time series at different times. Such correlations are usually examined using the autocorrelation although it shows basically the same information as the power spectrum. To understand the effect, we will first look at the models we already investigated.

Fig. 5.19A shows the autocorrelation of $Y$ for the model with continuous dynamics but control pulses at certain times $t_p$ defined by Eqns. (5.4) and (5.8). This model is equivalent to the minimal model (Eqns. 2.3 and 2.6) if only the times $t_p$ are considered. Hence, the autocorrelations for the minimal and the pulsed model look identical, but for the minimal model, there are no supporting points in between the times $t_p$. As we have seen for the minimal model in Chap. 2, values of $Y$ are correlated with the two preceding time steps. For lags of three steps and...
more, the time series is uncorrelated. This short range correlation is related to the slight dip in the power spectrum at high frequencies.

The continuous model shown in Fig. 5.19B, is still short-range correlated. The time constant for the exponential decay over few seconds is related to the first scaling regime in the power spectrum. Accordingly, it is mostly determined by the gain factor and memory length. A resonance peak in the spectrum is reflected by negative values of the autocorrelation. For the model with scaling noise, the exponential decay is slowed slightly. This is again consistent with the reduction of the first scaling exponent in the power spectrum reported before and therefore not shown. The second scaling regime is reflected by a very short exponential decay for small lags. This can hardly be seen here, but when plotting the log of the autocorrelation for short lags, there is a pronounced kink between 200 and 250 ms in the simulations and experimental time series which corresponds to the transition between the scaling regimes in the power spectrum.

An important characteristic of the volatility is, that it causes long-range autocorrelations in non-linear transforms of the considered variable like its absolute value, but not in the variable itself. Price changes are usually correlated for a few trading minutes, but their absolute values show at least two additional regimes.³

³In this thesis, the focus is put exclusively on the analysis of \( Y = |y| \) for the simulated time series because the absolute distances are analysed for the experimental time series. However, for the continuous model, the autocorrelation for \( y \) decays only marginally faster than for \( Y \). When scaling noise is included, the autocorrelation of \( y \) decays considerably faster within only one second and may become slightly negative. The autocorrelation of \( Y \) on the other hand decays slower than for the model with constant noise level. This is consistent with findings for the \( x \)- and \( y \)-components of the experimental data sets.
The first is a decay of correlations over few days while the second regime of correlations decays on time scales of the order of at least several hundred days. This long-range tail may consist of even more time scales than one.

To model volatility, processes with a time dependent variance are commonly studied in economics and finance. In the discrete-time autoregressive conditional heteroskedasticity (ARCH) models, the process has a gaussian conditional probability density and the variance depends on one or more previous values of the process. This process is generalised in GARCH processes where the variance also depends on its own previous values. Generally, processes in continuous time where the volatility is a random variable itself are called stochastic volatility (SV) models. Such models are very common in financial modelling.

While the dynamics of $\sigma(t)$ for the balancing model have been defined analogously to an ARCH process, it is of course possible to include dependencies on $\sigma(t)$ itself. Depending on the parameters, this can lead to slower dynamics of $\sigma(t)$ causing large clusters of fluctuations instead of short bursts.

Even for the simplest possible ARCH(1) model, the exact shape of the pdf is unknown, but highly parameter dependent. For the critical control model, the power law pdf is an intrinsic feature and the dynamics of $\sigma(t)$ can be almost arbitrary. These correlations then enter the model in addition to the short-range correlations that are always present in critical control. To demonstrate this possibility, Fig. 5.19C shows the continuous model where $\sigma(t)$ is defined by

$$\dot{\sigma}(t) = -\frac{1}{t_v}(\sigma_0 - \sigma(t)) + \sigma_y y(t - t_r)$$

(5.25)

which is a diffusion process that reverts to a stationary value $\sigma_0$ with a time constant $t_v$. Instead of an independent noise term, the process is driven by the fluctuations in the controlled variable $y$ scaled by some constant factor $\sigma_y$. Usually, these dynamics would be too simple to yield multiple time scales, but we are adding them to the time scales already present in the control model. This extension is one possible example for a critical control model that shows long-range correlations. Actually, the tail of the autocorrelation is qualitatively very similar to daily price changes for the New York Stock Exchange or the Dow Jones even when plotted in log-normal or log-log coordinates (not shown).

The volatility has a non-Gaussian pdf, but its shape is still under debate. Here, driving the volatility with $y$ causes it to be non-Gaussian distributed and correlated with $y$. As a recent example of a process with similar volatility dynamics the correlated exponential Ornstein Uhlenbeck (cexpOU) process can reproduce many stylised facts [32]. In this model, the logarithm of the volatility is a mean reverting diffusion process driven by a Gaussian random variable that is correlated with the one driving the main process. This yields multiple time scales and a log-normal distributed volatility.

As a final example of the simplicity of including arbitrary higher order correlations in the model, Fig. 5.19D shows the continuous model where $\sigma$ is modulated by a sinusoidal.
6. Summary and discussion

Human balancing behaviour exhibits fluctuations with statistical properties that are commonly associated with physical systems near critical points. Therefore, these fluctuations have been investigated using a combination of methods from statistical physics and control theory.

Critical control

Criticality emerges naturally in adaptive control by a very simple and intuitive mechanism: when fluctuations are compensated almost perfectly, only the intrinsic noise in the system is observable. A controller with strictly limited memory that continuously estimates the parameters of an unstable system also when reasonable balance is already achieved will lose its ability to perform appropriate control actions. This leads to huge fluctuations which allow the controller to observe the systems dynamics far away from the noise level. Then, a good estimation of the systems parameters may be obtained leading to small fluctuations again. This cycle between stability and instability turns the critical point into an attractor of the systems dynamics.

Existing explanations for critical fluctuations typically rely on either high-dimensional systems with non-linear interactions (Self-organized criticality) or fine-tuning a systems parameters to a stability boundary (intermittency). Critical control has the advantage that it allows to generate critical behaviour in simple systems without parameter tuning.

The minimal model for critical control is a simple stochastic map. A controller using optimal on-line estimation from only two past observations to predict the next state of a system can be derived from minimising the mean squared error. The resulting dynamics exhibits clear power law tails with an exponent $\delta = 2$ (Chap. 2).

Equation (2.8) reveals the mechanism behind this behaviour: the dynamical variable $y_i$ appears in the numerator of a multiplicative noise term. This causes a critical noise amplification if the controller has previously been successful in reducing the control error $y_i$ to near zero. The multiplicative noise results from optimal parameter estimation of a simple linearly unstable system with additive noise coupled to a controller.

Formally, this process is very similar to another multiplicative noise process called the Kesten process. This process is well known to produce a special kind of intermittency without saturation for extreme amplitudes. Power law distributed fluctuations can be generated, but require very precise parameter tuning.
Previous balancing models explicitly incorporated multiplicative noise to yield intermittent behaviour, required parameter tuning as well as some unrealistic assumptions about the system and did not yield power law distributed fluctuations in the controlled variable.

Critical control is also consistent with the current state of research in motor control in several ways. Most prominently, the CNS predicts future states of body and environment using forward models.

**Virtual balancing**

Virtual balancing provides a simple experimental paradigm to investigate human balancing behaviour. Data from such experiments was provided by Markus Riegel. Subjects were asked to balance an unstable target on a screen with a cursor controlled using a computer mouse. The experiments were designed to closely match the conditions of the theory. All data has been newly analysed using specially optimised methods (Chap. 3). For the experimental and simulated time series, small pdf exponents generally correspond to large fluctuations and vice versa.

For all subjects target-mouse distance distributions strongly deviate from Gaussians. The combined time series from all trials of each subject exhibit power law tails with exponents in the range of three to five \( \text{Tab. 5.1} \). Relative standard deviations of the fluctuations are several percent in daily data sets \( \text{Tab. 5.2} \) and two digit percentages for single trials \( \text{Tab. 5.3} \).

Power spectra are constant for low frequencies and show two regimes of power law scaling for high frequencies. The first exponent in the power spectra is between 0.5 and 1.7 and the second one between 2.4 and 3.7. Standard deviations are similar to those for the pdf, but the uncertainties due changes in the parameters for the analysis methods are even higher. The transition between both scaling regimes occurs around 4 to 5 Hz. The first regime is relatively straight while second regime is only approximately a power law and shows superimposed structures.

The scaling of the variance of the cumulated distances exhibits hyperdiffusion with an Hurst exponent that is constantly just below one for short times. Over time scales of few seconds, the time series become uncorrelated.

The minimal model does not reproduce the features of human motor control dynamics (Fig. 3.3). Correlations were very short, \( H = 0.5 \) for all times and the power spectrum didn’t exhibit two separate scaling-regions.

**Model Extensions**

The mechanism for generating power law distributed fluctuations in critical control is robust with respect to the introduction of a variety of model extensions. First, parameter estimation can be generalised to using \( m \geq 2 \) past timesteps. This yields integer tail exponents \( \delta_m = m \). The ad-hoc generalisation to an exponential decaying memory allows to adjust the tail exponent to continuous val-
At the beginning of research for this thesis, a new model was investigated that simply combined an exponential decaying memory and an estimator delay. This model can generate two scaling regimes in the power spectrum and Hurst exponents above 0.5 together with a power law tail in the pdf. The model has been published together with the newly analysed data in [8]. However, reproducing specific combinations of features observed experimentally is generally not possible. Also, relating time scales in the data with those in the model is difficult due to its discrete nature.

An extension to real reaction delays has been developed. Although it does not seem to provide significant benefits for the discrete model, reaction times are essential in continuous control. The important new aspect is, that the controller has to include its own previously planned actions in its forward model.

**Continuous control**

It has been shown, that critical control is possible in continuous time (Chap. 5). A generalisation starting from the discrete model is straightforward. Here, a finite reaction time during which control actions are performed is indispensable. Also, a gain for the control actions has to be included.

The continuous model can reproduce nearly all features found in the experimental data for a range of realistic parameters. Especially, the first scaling regime in the power spectrum is identified with the controllers actions. The transition to the second regime is determined by the reaction time. The second regime represents frequencies, that are above the controllers response. Here, the exponent is parameter independent around 2.5. The distance dynamics are correlated for seconds, with Hurst exponents constantly slightly below one for short times.

The best results are obtained, if the maximum precision of the controller is limited. The effective result is a noise level that increases when large fluctuations are observed. This prevents an exponential decay of large fluctuations due to the then precise information about the systems parameter. This model can reproduce the general shape of peaks and their clustering into bursts of the length of few seconds found in the experimental time series.

Realistic properties are obtained for a limited range of model parameters. The transition between the two scaling regimes around 4 to 5 Hz can be obtained with reaction times between 170 and 200 ms. This is consistent with the range of human reaction times (Sec. 1.2, Sec. 5.5). The other features hint on gain factors between one and two, probably closer to one and memory decay constants between 100 and 250 ms.

The only feature missing in the model as presented in this thesis are scaling exponents in the second regime of the power spectrum above three. These values can be observed for some subjects. A possible explanation for this effect are
passive damping properties of the system that are not included in the model. Accordingly, driving the model with lowpass filtered noise yields such exponents.

**Conclusions and outlook**

Critical control provides an intuitive and simple mechanism for systems that have a critical point as an attractor. It also provides a possible explanation for power law distributed fluctuations in human balancing behaviour. Starting from a simple random map, several extensions up to a continuous model are possible. The latter quantitatively reproduces nearly all characteristic features observed in time series obtained in virtual balancing experiments.

These results suggest, that the human nervous system may employ adaptive motor control using only a very limited memory of past observations for estimating the parameters of the controlled system. Intuitively, rapid estimation is essential in a non-stationary environment.

The fact that the consideration of a longer history yields faster decaying densities suggests, that power laws observed in sensor-motor control result from a compromise between stability and fast adaptation with respect to changes in system parameters. However, determining an optimal memory length depends on the exact kind of considered non-stationarity in a complex way. Future research may lead to a better understanding of these effects. It is, however, not inevitable, that the limitation of the memory is the consequence of an optimality principle purely based on the systems dynamics. For example, working memory limitations could simply restrict the length of the memory. Then, the critical fluctuating estimator would be the consequence of given biological restrictions.

Further questions regarding optimality principles include an optimal gain. Gains slightly above one may be the consequence of the subjects attempt to control as cautiously as possible. Also, limitations in the ability to move the mouse faster may play a role. On the other hand, it is also possible, that humans try to optimise the gain.

The continuous model has been derived with the aim to make it as similar to the existing discrete model as possible. The direct derivation of the continuous model using optimisation principles is another topic further investigations could focus on.

In the context of optimality, a comparison with the Kalman filter [33] could provide new insights. This filter is an optimal estimator for the state of stochastic, discrete time systems. It uses the difference between a predicted state and the actual measurement weighted with an optimal gain to iteratively improve its estimation. The filter may be derived for a systems whose state evolves according to Kestens process by using Bayes’ theorem. Humans and animals actually seem to behave as if they employ Bayesian estimators in many situations [34].

Critical control makes strong assumptions about memory limitations. Intuitively, it should be possible to use an estimator model to predict when the controller uses an unprecise parameter estimation and therefore has a high probabili-
ity to cause large fluctuations. In the basic model, Eq. 2.8 allows to reduce the insecurity about the next state of the system to be reduced to a Gaussian, thereby eliminating the power law tail. This approach is directly applicable to extended discrete models, but conceptional problems arise for continuous models that are unsolved as of this writing.

Due to the small number of parameters in the model, it should be possible to find optimal fits for a given time series using automatic optimisation algorithms like a gradient decent. Automatising the time-consuming process of fitting the model to a time-series would also allow to compare the fits for different subjects.

Future experiments could take the control models presented in this thesis to a critical test. Parameters like the reaction time may be measured directly by measuring the response to delta pulses applied to the distance between mouse and target. In simulations, averaging over the pulse responses allows to measure the reaction time. Also, if the pulse was strong enough to disturb the estimator, the shape of the response changes and characteristic differences occur if the estimator is delayed additionally to the reaction time. Such an extension has been presented for discrete models, but not for the continuous one. Although this extension is just as simple in the continuous case as it is in discrete time, no notable benefit could be found by introducing this additional parameter. Simulations indicate, that the response to changes in the system parameter $\alpha$ is too weak to be detected with available data set sizes. Accordingly, no such response could be found in existing time series.

Using higher sample rates could provide more information about the systems high frequency response. In simulations, this yields a longer high-frequency scaling regime and a Hurst exponent even closer to one (not shown).

Other possible variations of the experimental setup include delays or changes in the weight of the mouse or the friction of the surface. This could give insights into the influence of these factors that are not accounted for in the model. Also, changes in the parameter and effects of an varying parameter could be investigated more systematically without having different subjects balance under different conditions. Subjects could also be motivated by high scores to try to perform optimally.

Long term adaptation between trials or days could be investigated. In previous experiments, the number of trials per day was unfortunately not constant and experiments were not always performed on subsequent days. Observed variations in different subsets of existing data do not hint on changes in each subjects control behaviour that are clearly above the noise level. If long term adaptation is observed, switching between situations previously adapted to could provide insights into a possibly modular structure of adaptation.

A similar experimental paradigm for balancing would be to use a manipulandum. Then, the application of force fields would allow to investigate changes in open-loop control strategies like the overall muscle stiffness. Also, movements could be investigated in three dimensions or restricted to only one dimension. A manipulandum could also be used within a virtual reality setup. Displaying a
three dimensional simulation, the illusion of balancing a real stick could be created. This setup would still be simpler than capturing the motion of a real stick and offer more possibilities for variations. Such experiments could be compared with possible new critical control models that control the same simulation.

The concept of self-organized critical control may be applicable to other systems. Non-Gaussian fluctuations and similar scaling properties in the power spectrum were reported for the balancing of real sticks. This leads to the assumption, that humans may employ similar control strategies in both situations. There also seem to be similarities to postural control although the time series are anticorrelated over times scales of several seconds. For time scales below few seconds, positive correlations were reported and no correlations for longer times scales which is again consistent with balancing (Sec. 1.3.1).

Connections to completely different fields of research may be possible, too. As shown in Sec. 5.3 critical control models can be constructed with short-range and long-range correlations found in many complex systems. Even if the concept of a controller doesn’t hold in the particular system, the formal framework provides at least the potential to construct random processes with a variety of properties.

Financial markets may be taken as an example for other control systems that show critical fluctuations. More precisely, the returns or log-returns show fluctuation with a power law tail in the pdf. Returns are the price differences between subsequent observations of a price. A common explanation is the efficient market hypothesis. By exploiting opportunities to make profit, traders cause the system to gradually move into a direction that eliminates said opportunities. A market is called efficient if the available information is instantly processed and reflected in the prices of the goods that are traded. If there was a risk-free possibility to make profit using observed price changes, it would be instantly exploited and thereby eliminated. Therefore, price changes are indistinguishable from random fluctuations. Markets are usually regarded to be highly efficient [31].

The analogy to balancing models could be to consider the price dynamics an act of balancing to reach the optimal price. To do this, the market as a whole estimates the expected change in price and evolve accordingly. This estimation would be a rapid optimal estimation based on the currently available information. An new and important aspect of such a model would be, that local efficiency can actually lead to criticality. Generally, it is not even necessary that the controlled system is unstable. It is sufficient, that the controller tries to bring the deviation of the state variable to zero faster than the passive dynamics of the system.

Finally, the self-tuning of a controller to a critical state may be considered a special case of a macroscopic description for self organised criticality. Underneath this macroscopic control behaviour lies a microscopic system. In human balancing, this microscopic system is a neuronal network. In markets, the network consists of agents that trade in order to achieve profit. Open questions include the requirements on the network structure in order to perform critical control. Analogies between neurons exchanging action potentials and agents
trading goods thereby leading to a similar macroscopic behaviour of the system could be investigated. Another question is, in how far effective structures of a controller relate to distinctive entities in the structure of the underlying systems or emerge only in the macroscopic behaviour. Investigations could provide contributions to reduce the gap between the neuronal and behavioural level in brain research or between trading strategies and market dynamics.

The novel concept of self-organised critical control provides new connections between various fields research. Combining control theory and statistical physics reveals, that local optimisation can cause critical behaviour on a global scale. Critical fluctuations are found not only in balancing and financial time-series, but also in the distribution of earthquake magnitudes, velocity changes in turbulent fluids and many other systems. Critical control may contribute to a deeper understanding of the mechanisms behind these fluctuations.
A. Appendices

A.1. Abbreviations

Chap. chapter
Sec. section
Eq. equation
Fig. figure
pdf probability density function
iid independent identically distributed
const. constant
sde stochastic differential equation
CNS central nervous system
SOC self organised criticality

A.2. Additional figures

The following figures show the complete analyses for the combined experimental time series for each subject. This analysis includes the time series of the distances $Y = |y|$ versus iteration steps; the complementary cumulative distribution functions with a fit of the tail exponent $\delta$ and a gaussian with the same variance as the data for comparison (dotted line); the power spectrum with fits for the two scaling regimes and the variance of the cumulated distances over different time intervals with a fit of the scaling exponent which corresponds to two times the Hurst exponent $H$. The employed methods are described in Chap 3. Also, the complementary cumulative distribution functions and power spectra for the combined trials for each day and the single trials for each subject are shown. The results for the corresponding fits are listed in Tabs. 5.1 to 5.3.
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Figure A.1.: Analysis of the experimental data of subject 1a. Combined data: A: Time series; B: Complementary cumulative distribution function; C: Power spectrum; D: The variance of the cumulated magnitudes. Daily data: E: Complementary cumulative distribution function; F: Power spectrum. Single trials: G: Complementary cumulative distribution function; H: Power spectrum.
A.2. Additional figures

Figure A.2: Analysis of the experimental data of subject 1b. Combined data: 
A: Time series; B: Complementary cumulative distribution function; C: Power spectrum; D: The variance of the cumulated magnitudes. Daily data: 
A. Appendices

Figure A.3.: Analysis of the experimental data of subject 2. Combined data: A: Time series; B: Complementary cumulative distribution function; C: Power spectrum; D: The variance of the cumulated magnitudes. Daily data: E: Complementary cumulative distribution function; F: Power spectrum. Single trials: G: Complementary cumulative distribution function; H: Power spectrum.
Figure A.4: Analysis of the experimental data of subject 3. Combined data: A: Time series; B: Complementary cumulative distribution function; C: Power spectrum; D: The variance of the cumulated magnitudes. Daily data: E: Complementary cumulative distribution function; F: Power spectrum. Single trials: G: Complementary cumulative distribution function; H: Power spectrum.
Figure A.5.: Analysis of the experimental data of subject 4. Combined data: A: Time series; B: Complementary cumulative distribution function; C: Power spectrum; D: The variance of the cumulated magnitudes. Daily data: E: Complementary cumulative distribution function; F: Power spectrum. Single trials: G: Complementary cumulative distribution function; H: Power spectrum.
Figure A.6: Analysis of the experimental data of subject 5. Combined data: A: Time series; B: Complementary cumulative distribution function; C: Power spectrum; D: The variance of the cumulated magnitudes. Daily data: E: Complementary cumulative distribution function; F: Power spectrum. Single trials: G: Complementary cumulative distribution function; H: Power spectrum.
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Figure A.7: Analysis of the experimental data of subject 6. Combined data: A: Time series; B: Complementary cumulative distribution function; C: Power spectrum; D: The variance of the cumulated magnitudes. Daily data: E: Complementary cumulative distribution function; F: Power spectrum. Single trials: G: Complementary cumulative distribution function; H: Power spectrum.
A.2. Additional figures

Figure A.8: Analysis of the experimental data of subject 7. Combined data: A: Time series; B: Complementary cumulative distribution function; C: Power spectrum; D: The variance of the cumulated magnitudes. Daily data: E: Complementary cumulative distribution function; F: Power spectrum. Single trials: G: Complementary cumulative distribution function; H: Power spectrum.
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